TMA4195 Mathematical modelling 2005

Exercise set 7

Advice and suggestions: 2005–10–12

In an incompressible Newtonian fluid the density ρ is by definition constant. You may assume as known that

$$\nabla \cdot \mathbf{v} = 0 \tag{1}$$

in an incompressible fluid. Furthermore, such a fluid in a gravitational field is known to satisfy the *Navier–Stokes* equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{g} - \frac{\nabla p}{\rho} + \nu \nabla^2 \mathbf{v} \tag{2}$$

where $\mathbf{v}$ is the fluid velocity, ρ its density, and ν its kinematic viscosity. Also, p is pressure, which we may normalize to be zero at the surface of the fluid (since subtracting a constant from the pressure does not change the equations of motion).

In the rest of this problem you will use the lubrication theory approach to derive the equation for long waves in shallow water from (1) and (2). The waves are assumed to have lengths on the scale L, and the typical water depth is assumed to be H, where $H/L = \varepsilon \ll 1$. We work in two space dimensions with the y axis pointing up.

(a) Write $\mathbf{x} = (x^*, y^*)$ and $\mathbf{v} = (u^*, v^*)$. Write up (1) and (2) in component form, where time and pressure are written t^* and p^*. Introduce dimensionless variables using the scaling

$$x^* = Lx, \quad y^* = Hy, \quad u^* = Uu, \quad v^* = Vv, \quad t^* = Tt, \quad p^* = \rho g H p$$

What does (1) tell you about what the relationship between L, H, U, and V should be in a good scaling? Explain why the scales should satisfy $UT = L$ and $VT = H$, and why $p^* = \rho g H p$ is a good scaling. Introduce this into the equations, and explain why the scales should be picked so that $U^2 = gH$. Assume this is done, and express the resulting equations so the coefficients are expressed in terms of the parameters ε and $\mathrm{Re} = UH/\nu$. Estimate these two parameters for tidal waves in the North Sea ($H \approx 100\mathrm{m}$, $T \approx 6\mathrm{h}$, $\nu \approx 10^{-6}\mathrm{m}^2\mathrm{s}^{-1}$) – and discuss the usefulness of the system

$$u_t + uu_x + vu_y + p_x = 0, \quad p_y = -1, \quad u_x + v_y = 0 \tag{3}$$

Assume now that the ocean bottom is given by $y = 0$ and the surface by $y = h(x, t)$ (or $y^* = h^*(x^*, t^*)$ expressed in the original variables). The middle equation in (3) together with the boundary condition $p = 0$ for $y = h(x, t)$ gives $p = h(x, t) - y$.

(b) Show that this implies that if u is independent of y for $t = 0$ then the same holds for all $t > 0$. (Hint: Consider a fixed fluid particle – defined as a solution of $\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x}, t)$ – and compute $\mathrm{d}u/\mathrm{d}t$ for it).
Accordingly, explain how the first equation in (3) can be written as

$$u_t + uu_x + h_x = 0. \tag{4}$$

(c) Derive the formula for mass conservation:

$$h_t + (uh)_x = 0 \tag{5}$$

by integrating $u_x + v_y = 0$ over the area $x_1 < x < x_2$, $0 < y < h(x, t)$ and using Green's theorem. (Hint: A fluid particle which is once at the surface will always be there. This gives a boundary condition that you will need). Justify the use of the term *mass conservation* for this formula.

The two equations (4) and (5) together are known as the *shallow water equations.*