TMA4195 Mathematical modelling 2005

Exercise set 9

Advice and suggestions: 2005–11–09

This exercise is to be handed in by Friday, 18 November.
It will count as 10 % toward your final grade.

Problem 1: A popular foot race with many participants ("studentmila") happens in a stadium with a 400 m oval track.

(a) When lots of students are running in the same part of the track, they impede each other so that their speed decreases. A certain model leads to the partial differential equation on dimensionless form:

$$\rho_t + (1-2\rho)\rho_x = 0 \tag{1}$$

where $0 \le \rho \le 1$ and $\rho(x,t)$ is a 2π-periodic function of x. What are the assumptions of this model? Does it seem reasonable? (Compare with the traffic model.)

(b) Find (on *implicit* form) the exact solution of (1) given the initial condition

$$\rho(x,0) = \rho_0 + \varepsilon\cos x, \qquad 0 < \rho_0 \pm \varepsilon < 1,\ \varepsilon > 0, \tag{2}$$

valid at least for small t.

(c) Sketch the characteristics corresponding to the initial values of (2). Show that the smooth solution must break down after a while and develop a shock.

(d) When does the shock form, how does it move, and what happens as $t \to \infty$? (*Hint*: Consider the intersection of the characteristics starting at $(x,t) = (\frac{3}{2}\pi \pm \theta)$ where $0 < \theta < \pi$.) In hindsight, how could you have proceeded without this hint?

Problem 2: Solve the following singular perturbation problem for the function $y(x)$ to lowest order in ε, where $0 < \varepsilon \ll 1$:

$$\varepsilon y'' - yy' + 1 = 0, \qquad y(0) = 1,\ y(1) = 0.$$

There is a boundary layer at one end. Which one?