An impossible initial value problem

Harald Hanche-Olsen

hanche@math.ntnu.no

Consider the initial value problem for a function $x(t)$:

$$x\ddot{x} = 1, \qquad x(0) = 0, \quad \dot{x}(0) = a.$$

Since this is a second order ODE, you might expect to find a unique solution. However, the condition $x(0) = 0$ makes the equation *singular* at $t = 0$, so the usual existence theory breaks down, and the problem must be examined more closely.

Obviously, inserting the given initial conditions into the ODE at $t = 0$ produces the absurd equation $0 \cdot \ddot{x}(0) = 1$.

But we might still hope for a solution to $x\ddot{x} = 0$ for $t > 0$ which has a limit as $t \to 0$ satisfying the initial conditions. (If so, $\ddot{x} \to \infty$ as $t \to 0$.) We shall see that this is hopeless too.

Introduce the new dependent variable $y = \dot{x}$: The original equation becomes the system

$$\dot{x} = y, \quad \dot{y} = \frac{1}{x}, \qquad x(0) = 0, \quad y(0) = a.$$

Writing the system as $\mathrm{d}x = y\,\mathrm{d}t$, $\mathrm{d}y = x^{-1}\,\mathrm{d}t$ and dividing[1] we find the separable equation $\mathrm{d}x/\mathrm{d}y = xy$. The usual procedure for separable equations produces the general solution

$$x = Ae^{y^2/2}$$

where A is a *non-zero* constant ($x = 0$ does not satisfy the equation). The initial condition $x(0) = 0$ implies the impossible $A = 0$. In fact, for $A \neq 0$ the minimum value of $|x|$ is $|A|$, so we cannot even get $x = 0$ in the limit as $t \to 0$.

You will likely get a much more complicated equation for the problem of the modelling seminar, but still with the problematic leading term $x\ddot{x}$. The resulting initial value problem will be unsolvable, for much the same reason that this one is – though proving it is harder. What to do? Suspect that your model is wrong for small t, and think about possible ways to get around the difficulty.

[1] A formal, but correct procedure.