75042 Partial differential equations

Problem set #2 — suggested solutions

The first two problems this week were picked from the notes entitled "Elements of analysis".

Number 24: To show that a function f is *not* uniformly continuous, we must show that for *some* $\varepsilon > 0$ and *any* $\delta > 0$ there exists x and y with $|y - x| < \delta$ but $|f(y) - f(x)| > \varepsilon$.

First, $y^2 - x^2 = (x + y)(y - x)$. Given $\varepsilon > 0$ and $\delta > 0$, let $x = \varepsilon/\delta$ and $y = x + \delta/2$. Then $y^2 - x^2 > 2x \cdot \delta/2 = \varepsilon$.

Similarly, $1/x - 1/y = (y - x)/(xy)$. Given $\varepsilon > 0$ and $\delta > 0$, let $y = 2x$ where $x > 0$ is yet to be determined. Then $y - x = x$ and $1/x - 1/y = 1/(2x)$. If $0 < x < \min(\delta, 1/(2\varepsilon))$ we get $y - x < \delta$ and $1/x - 1/y > \varepsilon$.

For any differentiable real function f on some interval, the secant theorem states $f(y) - f(x) = f'(\xi) \cdot (y - x)$ for some ξ between x and y. In particular $|f(y) - f(x)| \le \sup_\xi |f'(\xi)| \cdot |y - x|$. When $f(x) = \arctan x$ we have $|f'(\xi)| = 1/(1 + \xi^2) \le 1$, so $|\arctan y - \arctan x| \le |y - x|$. Thus arctan is actually Lipschitz continuous, and therefore certainly uniformly continuous.

Number 27: Assume that, for every $\varepsilon > 0$, there is some $\delta > 0$ so that

$$\left|\frac{\partial f}{\partial x}(x, y) - \frac{\partial f}{\partial x}(x_0, y)\right| < \varepsilon g(y)$$

where $\int_{-\infty}^{\infty} g(x, y)\,\mathrm{d}y < \infty$. Let $F(x) = \int_{-\infty}^{\infty} f(x, y)\,\mathrm{d}y$. Then as in the proof of Proposition 26, we get

$$\begin{aligned}
\left|\frac{1}{x - x_0}\big(F(x) - F(x_0)\big) - \int_{-\infty}^{\infty} \frac{\partial f}{\partial x}(x_0, y)\,\mathrm{d}y\right| &= \left|\frac{1}{x - x_0}\int_{-\infty}^{\infty}\int_{x_0}^{x}\left(\frac{\partial f}{\partial x}(\xi, y) - \frac{\partial f}{\partial x}(x_0, y)\right)\,\mathrm{d}\xi\,\mathrm{d}y\right| \\
&\le \frac{1}{|x - x_0|}\int_{-\infty}^{\infty}\int_{x_0}^{x}\left|\frac{\partial f}{\partial x}(\xi, y) - \frac{\partial f}{\partial x}(x_0, y)\right|\,\mathrm{d}\xi\,\mathrm{d}y \\
&< \frac{1}{|x - x_0|}\int_{-\infty}^{\infty}\int_{x_0}^{x} \varepsilon g(y)\,\mathrm{d}\xi\,\mathrm{d}y \\
&= \varepsilon \int_{-\infty}^{\infty} g(y)\,\mathrm{d}y
\end{aligned}$$

and since the integral is just a finite constant, the desired differentiability result follows. Of course, in order for the above calculations to be correct, we also had to assume that the improper integrals $\int_{-\infty}^{\infty} f(x, y)\,\mathrm{d}y$ and $\int_{-\infty}^{\infty} f_y(x, y)\,\mathrm{d}y$ both converge.

Laplacian in the disk: An unusual but quick way to derive the Laplacian in polar coordinates, is the following: Apply the divergence theorem to Du in the region $r_1 < r < r_2$, $\theta_1 < \theta < \theta_2$:

$$\begin{aligned}
\iint_R \Delta u\,\mathrm{d}x\mathrm{d}y &= \int_{r_1}^{r_2} \frac{-u_\theta(r, \theta_1)}{r}\,\mathrm{d}r + \int_{\theta_1}^{\theta_2} u_r(r_2, \theta)\, r_2\mathrm{d}\theta + \int_{r_1}^{r_2} \frac{u_\theta(r, \theta_2)}{r}\,\mathrm{d}r + \int_{\theta_1}^{\theta_2} -u_r(r_1, \theta)\, r_1\mathrm{d}\theta \\
&= \int_{r_1}^{r_2}\int_{\theta_1}^{\theta_2}\left((ru_r)_r + \frac{u_{\theta\theta}}{r}\right)\,\mathrm{d}\theta\,\mathrm{d}r
\end{aligned}$$

which, when compared to the formula $\iint_R \Delta u\,\mathrm{d}x\mathrm{d}y = \int_{r_1}^{r_2}\int_{\theta_1}^{\theta_2} \Delta u(r, \theta)\,\mathrm{d}\theta\, r\mathrm{d}r$ yields the inevitable conclusion

$$\Delta u = \frac{(ru_r)_r}{r} + \frac{u_{\theta\theta}}{r^2}.$$

Attempting $u(r, \theta) = R(r)\Theta(\theta)$ we immediately arrive at $(rR')'\Theta/r + R\Theta''/r^2 = 0$ which we multiply by $r^2/(R\Theta)$ to get $r(rR')'/R + \Theta''/\Theta = 0$, so that $r(rR')'/R = -\Theta''/\Theta$ is constant. Since Θ must be periodic with period 2π this constant must be of the form n^2 with n an integer, and we may write $\Theta = e^{in\theta}$. The R equation may be written $r(rR')' = n^2 R$, which has the solutions (linear combinations of) $R = r^{\pm n}$. We want our solutions to be continuous at the origin, so from these two we pick only $R = r^{|n|}$. (For $n = 0$ the general solution is $R = A + B\ln r$, and again we must throw away the part that is singular at the origin, leaving $R = 1 = r^{|0|}$.)

Taking linear combinations of the product solutions, we are therefore led to trying a series

$$u(x,y)=\sum_{n=-\infty}^{\infty} c_n r^{|n|}e^{in\theta}$$

where $x=r\cos\theta$ and $y=r\sin\theta$ as usual. Clearly, each term in the series is harmonic. When we set $r=1$ in the series and use the boundary conditions $u(\cos\theta,\sin\theta)=g(\theta)$, we simply get the Fourier series for g. Thus we should have

$$c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\varphi)e^{-in\varphi}\,\mathrm{d}\varphi.$$

We insert this back into the series for u, and interchange the order of summation and integration to get

$$u(x,y)=\int_{-\pi}^{\pi} g(\varphi)P(r,\theta-\varphi)\,\mathrm{d}\varphi$$

where P is the famous *Poisson kernel*:

$$\begin{aligned}P(r,\theta)&=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta}=\frac{1}{2\pi}\left(\sum_{n=0}^{\infty} r^n e^{in\theta}+\sum_{n=0}^{\infty} r^n e^{-in\theta}-1\right)\\&=\frac{1}{2\pi}\left(\frac{1}{1-re^{i\theta}}+\frac{1}{1-re^{-i\theta}}-1\right)=\frac{1}{2\pi}\frac{1-r^2}{1+r^2-2r\cos\theta}.\end{aligned}$$

As long as g is integrable ($\int_{-\pi}^{\pi}|g(\theta)|\,\mathrm{d}\theta<\infty$) we may differentiate under the integral sign in the integral formula for u above, yielding

$$\Delta u=\int_{-\pi}^{\pi} g(\varphi)\left(\frac{(rP_r(r,\theta-\varphi))_r}{r}-\frac{P_{\theta\theta}(r,\theta-\varphi)}{r^2}\right)\mathrm{d}\varphi=0.$$

(I verified that the integrand is zero using Maple; there is a worksheet available from my Web page which you may download and load into `xmaple` if you're curious.)

As for the boundary conditions, note that $P(r,\theta)\to 0$ uniformly in $\varepsilon<\theta\le\pi$ for every $\varepsilon>0$ as $r\to 1$, and also that $\int_{-\pi}^{\pi} P(r\theta)\,\mathrm{d}\theta=1$ for all r (most easily seen from the series representation for P). These are properties typical of an approximate delta function; we'll get back to those.

Problem set #3 — for next week

#3.1: Evans problem 1 (p. 80).

#3.2: If f is continuous at x, prove that $\displaystyle⨍_{\partial B(x,\varepsilon)} f\,\mathrm{d}x\to f(x)$ when $\varepsilon\searrow 0$.

#3.3: Evans problem 3 (p. 80).

#3.4: Let u be a C^2 function on some open set. Prove that u cannot have a local maximum at any point x where $\Delta u(x)>0$.

#3.5: Let Ω be a bounded open set, and let $u\in C^2(\Omega)\cap C(\overline{\Omega})$ satisfy the equation $\Delta u=u^3-u$ in Ω. Assume that $u=0$ on $\partial\Omega$. Show that $-1\le u\le 1$ throughout Ω.

#3.6: Evans problem 4 (p. 81).