75042 Partial differential equations

Problem set #4 — suggested solutions

#4.1: By Harnack's inequality, if $\tilde{K} \subset \Omega$ is a *connected* compact set then there is a constant C so that $\max_{\tilde{K}} u \le C \min_{\tilde{K}} u$ whenever $u > 0$ is harmonic in Ω.

It seems quite plausible that, given $x \in \Omega$ and a compact $K \subset \Omega$, there exists a *connected* compact set $\tilde{K} \subset \Omega$ with $K \cup \{x\} \subset \tilde{K}$. Let C correspond to this set $\tilde{K}$ like in the previous paragraph. Then, if (u_n) is a sequence of positive harmonic functions on Ω and $x \in \Omega$ with $u_n(x) \to 0$, we find

$$\max_K u_n \le \max_{\tilde{K}} u_n \le C \min_{\tilde{K}} u_n \le u_n(x) \to 0 \qquad (n \to \infty)$$

and so $u_n|_K \to 0$ uniformly.

It remains to prove the "plausible" statement of the previous paragraph. First, replace K by $K \cup \{x\}$. If K has only a finite number of components, it is easy: Just make a path between each pair of components and let $\tilde{K}$ be the union of K and all these paths. However K may have infinitely many components. Our first task, then, is to extend K to a compact subset of Ω with finitely many components. But the open balls $U(x, \delta(x))$ where $x \in K$ and $\delta(x) = \operatorname{dist}(x, \partial\Omega)/2$ form an open cover of K, and so by compactness a finite number of these also cover K. Taking closures, we get

$$K \subset \bigcup_{i=1}^{m} B(x_i, \delta(x_i))$$

Now, for $i = 1, 2, \dots, m-1$ let γ_i be a path in Ω from x_i to x_{i+1} (here we use the connectedness of Ω); i.e., $\gamma_i \colon [0,1] \to \Omega$ is continuous with $\gamma_i(0) = x_i$ and $\gamma_i(1) = x_{i+1}$. Now let

$$\tilde{K} = \bigcup_{i=1}^{m} B(x_i, \delta(x_i)) \cup \bigcup_{i=1}^{m-1} \gamma_i[0,1]$$

which is connected, compact, and $K \cup \{x\} \subset K \subset \Omega$.

#4.2: If u is bounded and harmonic in $\mathbb{R}^n$ then $u + b \ge 0$ in $\mathbb{R}^n$ if b is a lower bound for u. To prove that u is constant is clearly equivalent with proving $u + b$ constant. Thus we will replace u by $u + b$, and therefore assume $u \ge 0$ throughout.

Fix two points x and y in $\mathbb{R}^n$ and set $a = |x - y|$. The inclusion $B(y, r) \subset B(x, r + a)$ together with the mean value theorem proves

$$\begin{aligned} u(x) &= \fint_{B(x,r+a)} u(z)\,\mathrm{d}z = \frac{1}{\alpha(n)(r+a)^n} \int_{B(x,r+a)} u(z)\,\mathrm{d}z \\ &\ge \frac{1}{\alpha(n)(r+a)^n} \int_{B(y,r)} u(z)\,\mathrm{d}z = \frac{\alpha(n)r^n}{\alpha(n)(r+a)^n} \fint_{B(y,r)} u(z)\,\mathrm{d}z \\ &= \left(\frac{r}{r+a}\right)^n u(y) \to u(y) \quad \text{as } r \to \infty. \end{aligned}$$

Thus $u(x) \ge u(y)$. Exchanging x and y we get the opposite inequality, and so $u(x) = u(y)$. Therefore u is constant.

We have only used the existence of a *lower* bound on u: So a harmonic function which is bounded below (or above) on $\mathbb{R}^n$ is constant.

We may get the same conclusion just based on a suitable growth condition on u. Subtracting the two sides of the inequality in the above computation we have

$$\frac{1}{\alpha(n)(r+a)^n} \int_{B(x,r+a)\setminus B(y,r)} u(z)\,\mathrm{d}z \ge 0$$

but the same conclusion will hold if the lefthand side is greater than some quantity which tends to 0 as $r \to \infty$. This will hold if $|u(z)| = o(|z|)$ as $|z| \to \infty$. (Note that the domain of integration in the above integral is contained in the region $\{z \colon r - |y| \le |z| \le r + |x| - a\}$ to see this. This result is best possible, since $u(x) = x_1$ is an example of a nonconstant harmonic function.)

#4.3: The standard separation of variables procedure tells us to look for solutions to $\Delta u = 0$ on the form $u(x,y) = X(x)Y(y)$ with $Y(0) = Y(\pi) = 0$. The PDE becomes $X''/X + Y''/Y = 0$ (after dividing by XY), so each term must be constant, and we get $Y(y) = \sin ky$ ($k = 1, 2, \ldots$) because of the conditions on Y. Then $X(x) = Ae^{kx} + Be^{-kx}$, and we get unbounded solutions. The example tells us that the maximum principle may fail for unbounded domains.

#4.4: When $|x| < r$ and $|y| = r$ we get $r - |x| \le |y - x| \le r + |x|$ (by the triangle inequality).

Poisson's formula (Evans formula (45) on p. 43) gives us (and here we must rely on $g \ge 0$ together with the above inequality, plus the identity $r^2 - |x|^2 = (r - |x|)(r + |x|)$):

$$\begin{aligned} u(x) &= \frac{r^2 - |x|^2}{n\alpha(n)r} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n} \,\mathrm{d}S(y) \le \frac{r^2 - |x|^2}{n\alpha(n)r} \int_{\partial B(0,r)} \frac{g(y)}{(r - |x|)^n} \,\mathrm{d}S(y) \\ &= \frac{(r + |x|)n\alpha(n)r^{n-1}}{n\alpha(n)r(r-|x|)^{n-1}} \fint_{\partial B(0,r)} g(y)\,\mathrm{d}S(y) = r^{n-2} \frac{r + |x|}{(r - |x|)^{n-1}} u(0) \end{aligned}$$

which is what we wanted. The other inequality is shown similarly, using the other half of the inequality of the first paragraph.

#4.5: If u is subharmonic and v is harmonic, then $u - v$ is subharmonic. By Evans problem 4b (p. 81, our problem #3.6) then $\max_{\overline{\Omega}}(u - v) \le \max_{\partial\Omega}(u - v) \le 0$, so $u \le v$ in Ω.

#4.6: We take the hint, and determine constants a and b so that $M(r_i) = a \ln r_i + b$ for $i = 1, 2$. Then $u(x,y) \le a \ln r_i + b$ when $x^2 + y^2 = r_i^2$; that is, $u \le v$ on the boundary of the annulus, where $v(x,y) = a \ln \sqrt{x^2 + y^2} + b$. By the previous exercise then $u \le v$ in the entire annulus, and so $M(r) \le a \ln r + b$.

It only remains to determine a and b, which are given by a pair of linear equations:

$$\begin{aligned} \begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} \ln r_2 & 1 \\ \ln r_1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} M(r_2) \\ M(r_1) \end{bmatrix} = \frac{1}{\ln r_2 - \ln r_1} \begin{bmatrix} 1 & -1 \\ -\ln r_1 & \ln r_2 \end{bmatrix} \begin{bmatrix} M(r_2) \\ M(r_1) \end{bmatrix} \\ &= \frac{1}{\ln(r_2/r_1)} \begin{bmatrix} M(r_2) - M(r_1) \\ -M(r_2)\ln r_1 + M(r_1)\ln r_2 \end{bmatrix} \end{aligned}$$

and inserting these values of a and b into $M(r) \le a \ln r + b$ we get the desired inequality.

Problem set #5 — for next week

#5.1: Evans problem 9 (pp. 81–82).

#5.2: Evans problem 11 (p. 82).

#5.3: Evans problem 13 (p. 82).

#5.4: Evans problem 14 (p. 82).

#5.5: Evans problem 16 (p. 83).