75042 Partial differential equations

Problem set #5 — suggested solutions

#5.1: In case you got nowhere on this, here is a hint: Consider v on the boundary of the ball and apply the Poisson integral formula. Maybe you should try again? The full solution is at the end.

#5.2: That $u_\lambda(x,t) = u(\lambda x, \lambda^2 t)$ solves $u_t - \Delta u = 0$ if u does so, is trivial, since the chain rule yields $u_{\lambda,t} - \Delta u_\lambda = \lambda^2(u_t - \Delta u)$.

(In cases like this where not all subscripts mean differentiation, is is handy to introduce a comma to separate those subscripts that indicate a derivative from those that do not.)

Now differentiate u_λ with respect to λ. Equality of mixed partials, i.e., $u_{\lambda,\lambda t} = u_{\lambda,t\lambda}$ and $\Delta u_{\lambda,\lambda} = (\Delta u_\lambda)_\lambda$ guarantee that $u_{\lambda,\lambda}$ satisfies the heat equation. Put $\lambda = 1$ to obtain the solution $v = x \cdot Du + 2tu_t$.

#5.3: With $v(x,t) = e^{ct}u(x,t)$ we see that v satisfies the equation $v_t - \Delta v = e^{ct} f$ with the initial value $v(x,0) = g(x)$. There is an explicit formula for the solution to this problem on the top of p. 53. Just replace u by v and $f(y,s)$ by $e^{cs}f(y,s)$ in that formula, and finally set $u(x,t) = e^{-ct}v(x,t)$. I omit the full detailed formula here.

#5.4: Extend $v(x,t) = u(x,t) - h(t)$ to $x < 0$ by odd reflection: $v(-x,t) = -v(x,t)$. Clearly, v is continuous at $x = 0$ (because of the boundary condition $u(0,t) = h(t)$). When $x > 0$ we find $v_t - v_{xx} = u_t - u_{xx} - h'(t) = -h'(t)$. The same relation, with a sign reversal, will hold for $x < 0$ by antisymmetry, so we have

$$v_t - v_{xx} = \begin{cases} -h'(t) & x > 0, \\ h'(t) & x < 0. \end{cases}$$

Now, the solution to the problem $v_t - v_{xx} = f(x,t)$ with $v(x,0) = 0$ is given by Duhamel's principle (where we replace $f(y,s)$ by the righthand side of the above equation of course):

$$\begin{aligned}
v(x,t) &= \int_0^t \frac{1}{\sqrt{4\pi(t-s)}} \int_{\mathbb{R}} \exp\left[-\frac{(x-y)^2}{4(t-s)}\right] f(y,s)\,\mathrm{d}y\,\mathrm{d}s \\
&= \int_0^t \frac{1}{\sqrt{4\pi(t-s)}} \left(\int_{-\infty}^0 - \int_0^\infty\right) \exp\left[-\frac{(x-y)^2}{4(t-s)}\right] \mathrm{d}y\, h'(s)\,\mathrm{d}s \\
&= \int_0^t \underbrace{\frac{1}{\sqrt{\pi}} \left(\int_{x/\sqrt{4(t-s)}}^\infty - \int_{-\infty}^{x/\sqrt{4(t-s)}}\right) \exp[-z^2]\,\mathrm{d}z}_{k(s)}\; h'(s)\,\mathrm{d}s \\
&= k(t)h(t) - k(0)h(0) - \int_0^t k'(s)h(s)\,\mathrm{d}s
\end{aligned}$$

where we note that $k(t) = 0$, $h(0) = 0$, and $k'(s)$ is easily computed as

$$k'(s) = -\frac{x}{\sqrt{4\pi}(t-s)^{3/2}} \exp\left[-\frac{x^2}{4(t-s)}\right].$$

From this we obtain the required formula, which is as shown in the problem with the term $h(t)$ added (clearly there is a misprint).

#5.5: It is probably a good idea to extend our given function $g \in C([0,1])$ to have compact support, say $g(x) = 0$ when $x \notin [-1,2]$, and let g vary linearly from 0 to $g(0)$ on the interval $[-1,0]$ and linearly from $g(1)$ to 0 on $[1,2]$. We let

$$u(x,t) = \int_{\mathbb{R}} \Phi(y,t)g(x-y)\,\mathrm{d}y = \int_{x-2}^{x+1} \Phi(y,t)g(x-y)\,\mathrm{d}y \tag{1}$$

for $t > 0$, and $u(x,0) = g(x)$. Here Φ is the standard heat kernel. That u does in fact solve the heat equation is of no use to us here, but its continuity in $t \geq 0$ is useful: Not only does $u(x,t) \to g(x)$

as $t \searrow 0$, but u is uniformly continuous on every compact subset of the closed upper half plane $t \geq 0$, and therefore the function $x \mapsto u(x,t)$ converges uniformly to g on every bounded interval — and in particular it does so on $[0,1]$.

In other words, given $\varepsilon > 0$ we can find $\delta > 0$ so that $|u(x,t) - g(x)|$ whenever $0 \leq x \leq 1$ and $0 < t < \delta$. From now on, we fix one such t. It only remains to approximate $u(x,t)$ by a polynomial.

Now we use the explicit form of the heat kernel:

$$\Phi(y,t) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{y^2}{4t}\right)$$

The plan is to replace the exponential in Φ by a partial sum of the Taylor series $\exp(z) = \sum_{k=0}^{\infty} z^k/k!$. We are interested in $u(x,t)$ for $x \in [0,1]$. We see that the integral in (1) only uses values of $\Phi(y,t)$ with $|y| \leq 3$, which implies $y^2/(4t) \leq 9/(4t)$.

For our particular fixed t, let n be so large that $|e^z - \sum_{k=0}^{n} z^k/k!| < \varepsilon\sqrt{4\pi t}$ whenever $|z| \leq 9/(4t)$ (just make sure that $\sum_{k=n+1}^{\infty} 9^k/(k!(4t)^k) < \varepsilon\sqrt{4\pi t}$). Let

$$\Psi(y,t) = \frac{1}{\sqrt{4\pi t}} \sum_{k=0}^{n} \frac{y^{2k}}{k!(4t)^k}$$

so that $|\Psi(y,t) - \Phi(y,t)| < \varepsilon$ whenever $|y| \leq 3$ (and t is still fixed). Let

$$P(x) = \int_{x-2}^{x+1} \Psi(y,t) g(x-y)\,\mathrm{d}y = \int_{-1}^{2} \Psi(x-y,t) g(y)\,\mathrm{d}y$$

P is a polynomial, for $\Psi(x-y,t)$ is a polynomial of degree $2n$ in $x-y$, and so differentiation of the above integral more than $2n$ times yields zero. Whenever $x \in [0,1]$ we get $|g(x) - P(x)| \leq |g(x) - u(x,t)| + |u(x,t) - P(x)| < \varepsilon + 4\varepsilon \max|g|$ which shows that P can be made to approximate g arbitrarily well in the uniform norm.

#5.1: Extend u to a function v on the full ball by antisymmetry in the x_n variable, as explained in the problem. The function

$$w(x) = \int_U K(x,y) v(y)\,\mathrm{d}S(y)$$

is then continuous in $\overline{U}$ (v is certainly continuous on ∂U, since $u = 0$ on $x_n = 0$) and harmonic in U. The Poisson kernel $K(x,y)$ is clearly symmetric in y_n when $x_n = 0$ (see its formula!) and therefore, when $x_n = 0$ the integrand above is antisymmetric in y_n. Thus $w(x) = 0$ when $x_n = 0$. Since w is harmonic in U^+ with the same boundary values as u, $w = u$ in all of U^+. Further, $w(x)$ is antisymmetric in x_n and therefore $w = v$. This antisymmetry can be seen directly (note that $K(x,y)$ in unchanged if x_n and y_n both are replaced by $-x_n$ and $-y_n$ respectively).

Alternatively, one can note that the function $-w(x_1, \ldots, x_{n-1}, -x_n)$ is harmonic with the same boundary values as w to get the antisymmetry (and in particular the vanishing of $w(x)$ when $x_n = 0$).

Problem set #6 — for next week

#6.1: Evans problem 22 (p. 84).

#6.2: We say a continuous function u is *subharmonic* in an open set Ω if $u(x) \leq \fint_{\partial B(x,r)} u\,\mathrm{d}S$ whenever $B(x,r) \subset \Omega$. Note that we do not require u to be differentiable. Prove that, if u is subharmonic and v is harmonic in Ω and both functions are continuous on $\overline{\Omega}$ with $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω. Also prove that, if $u \in C^2(\Omega)$, then u is subharmonic in the present sense if and only if $-\Delta u \leq 0$ in Ω.