75042 Partial differential equations

Problem set #6 — suggested solutions

#**6.1:** From d'Alembert's formula

$$u(x,t) = \frac{1}{2}\big(g(x+t) + g(x-t)\big) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,\mathrm{d}y$$

we immediately get

$$u_t = \frac{1}{2}\big(g'(x+t) - g'(x-t) + h(x+t) + h(x-t)\big) = F(x+t) - G(x-t),$$
$$u_x = \frac{1}{2}\big(g'(x+t) + g'(x-t) + h(x+t) - h(x-t)\big) = F(x+t) + G(x-t)$$

where

$$F = \frac{1}{2}(g' + h) \quad \text{and} \quad G = \frac{1}{2}(g' - h).$$

Substituting this into the kinetic and potential energies

$$k(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_t^2\,\mathrm{d}x \quad \text{and} \quad p(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_x^2\,\mathrm{d}x$$

and then adding, we get

$$\begin{aligned} k(t) + p(t) &= \frac{1}{2}\int_{-\infty}^{\infty} \Big[\big(F(x+t) - G(x-t)\big)^2 + \big(F(x+t) + G(x-t)\big)^2\Big]\mathrm{d}x \\ &= \int_{-\infty}^{\infty} F(x+t)^2\,\mathrm{d}x + \int_{-\infty}^{\infty} G(x-t)^2\,\mathrm{d}x = \int_{-\infty}^{\infty} F(x)^2\,\mathrm{d}x + \int_{-\infty}^{\infty} G(x)^2\,\mathrm{d}x \end{aligned}$$

which is clearly independent of t. Similarly, subtracting we get

$$\begin{aligned} k(t) - p(t) &= \frac{1}{2}\int_{-\infty}^{\infty} \Big[\big(F(x+t) - G(x-t)\big)^2 - \big(F(x+t) + G(x-t)\big)^2\Big]\mathrm{d}x \\ &= -2\int_{-\infty}^{\infty} F(x+t)G(x-t)\,\mathrm{d}x \end{aligned}$$

which is zero for large t, since F and G both have compact support and therefore $F(x+t)G(x-t) = 0$ for all x when t is sufficiently large.

#**6.2:** It is helpful to note that a subharmonic function u (in the present sense) satisfies $u(x) \le \fint_{B(x,r)} u\,\mathrm{d}y$ when $B(x,r) \subset \Omega$, as we can deduce from integration in "polar" coordinates:

$$\fint_{B(x,r)} u\,\mathrm{d}y = \frac{1}{\alpha(n)r^n}\int_0^r \int_{\partial B(x,\rho)} u\,\mathrm{d}S\,\mathrm{d}\rho \ge \frac{1}{\alpha(n)r^n}\int_0^r n\alpha(n)\rho^{n-1}u(x)\,\mathrm{d}\rho = u(x)$$

If u is subharmonic and v is harmonic then $w = u - v$ is subharmonic, since v satisfies the mean value property. Also $w \le 0$ on $\partial\Omega$, and we have to prove that $w \le 0$ in Ω.

Now a subharmonic function satisfies the strong maximum principle, for if w achieves its maximum M at a point $x \in \Omega$ then

$$M = w(x) \le \fint_{B(x,r)} w\,\mathrm{d}y \le M$$

where the second inequality becomes strict if $w < M$ anywhere within $B(x,r)$. That is obviously absurd, so w must equal M in all of $B(x,r)$. It follows, just as in the proof of the maximum principle for harmonic functions, that u must be constant on the component of Ω containing x.

From the strong maximum principle follows the weak maximum principle, that is w achieves its maximum on $\partial\Omega$. Since $w \le 0$ there, $w \le 0$ everywhere — and so we have proved $u \le v$ in Ω.

For the last question, note that if $u \in C^2(\Omega)$ then

$$\varphi(r) = \fint_{\partial B(x,r)} u \, dS \quad \text{yields} \quad \varphi'(r) = \frac{r}{n} \fint_{B(x,r)} \Delta u \, dy$$

(see the proof of the mean value formula on p. 30 of Evans' notes.)

Clearly, if $-\Delta u \le 0$ in Ω we get $\varphi' \ge 0$, and since $\varphi(r) \to u(x)$ when $r \searrow 0$, we find $\varphi(r) \ge u(x)$ — and so u is subharmonic.

On the other hand, if $-\Delta u > 0$ at some point in Ω the inequality holds in an open set, since Δu is continuous. But if we place x in this open set, $\varphi'(r) < 0$ for small r, and hence $\varphi(r) < u(x)$ for small r. Hence u is not subharmonic.

We conclude that, for C^2 functions, the new and old definitions of "subharmonic" coincide.

Problem set #7 — for next week

#**7.1:** Evans problem 20 (p. 84).

#**7.2:** Evans problem 23 (p. 84).

#**7.3:** Let A be a bounded, open subset of $\mathbb{R}^n$, and let $u\colon A \to \mathbb{R}^m$ be an integrable function. Let $\varphi\colon \mathbb{R}^m \to \mathbb{R}$ be convex. Prove *Jensen's inequality*:

$$\varphi\Big(\fint_A u(x) \, dx \Big) \le \fint_A \varphi(u(x)) \, dx.$$

(*Hint*: Let $\bar{u} = \fint_A u(x) \, dx$. By the convexity of φ, there exists some $a \in \mathbb{R}^m$ so that $\varphi(v) \ge \varphi(\bar{u}) + a \cdot (v - \bar{u})$ for all $v \in \mathbb{R}^m$. This is true even if φ is not differentiable at $\bar{u}$.)

Explain why the same result holds if the integral is replaced by a surface integral over some surface of finite area.

#**7.4:** Prove that, if $u\colon \Omega \to \mathbb{R}^m$ is harmonic (i.e., each component function is harmonic) and $\varphi\colon \mathbb{R}^m \to \mathbb{R}$ is convex (but not necessarily differentiable) then $\varphi \circ u$ is subharmonic (as defined in problem #6.2). The composition $\varphi \circ u$ is defined by $\varphi \circ u(x) = \varphi(u(x))$.