75042 Partial differential equations

Problem set #7 — suggested solutions

#7.1: Unfortunately, there was an error in the problem. With $U(x;r,t) = \fint_{\partial B(x,r)} u(y,t)\,\mathrm{d}t$ we should set $\tilde U(x;r,t) = rU(x;r,t)$ (and *not* $\tilde U = (rU)_r/r$ as suggested in the problem).

This gives us $\tilde U_{rr} = (rU)_{rr} = (U + rU_r)_r = 2U_r + rU_{rr}$, and of course $\tilde U_{tt} = rU_{tt}$.

For $n = 3$, the Euler–Poisson–Darboux equation [(12) on p. 69 in Evans] is $U_{tt} = U_{rr} + 2U_r/r$. Multiplying this by r produces the wave equation $\tilde U_{tt} = \tilde U_{xx}$.

We are trying to solve $u_{tt} - \Delta u = 0$ in $t > 0$, $x \in \mathbb{R}^3$ with the initial values $u(x,0) = g(x)$ and $u_t(x,0) = h(x)$. By setting $G(x;r) = \fint_{\partial B(x,r)} g\,\mathrm{d}S$, $H(x;r) = \fint_{\partial B(x,r)} h\,\mathrm{d}S$, $\tilde G(x;r) = rG(x;r)$, and $\tilde H(x;r) = rH(x;r)$ we get an initial value problem for $\tilde U$ which is solved by the d'Alembert formula:

$$\tilde U(x;r,t) = \frac12\big(\tilde G(x;r+t) + \tilde G(x;r-t)\big) + \frac12\int_{r-t}^{r+t} \tilde H(x;\rho)\,\mathrm{d}\rho.$$

Simply divide by r to get

$$\begin{aligned}
U(x;r,t) &= \frac12\Big(\frac{r+t}{r}G(x;r+t) + \frac{r-t}{r}G(x;r-t)\Big) + \frac{1}{2r}\int_{r-t}^{r+t} \rho H(x;\rho)\,\mathrm{d}\rho \\
&= \frac{G(x;r+t) + G(x;t-r)}{2} + t\,\frac{G(x;t+r) - G(x;t-r)}{2r} + \frac{1}{2r}\int_{t-r}^{t+r} \rho H(x;\rho)\,\mathrm{d}\rho
\end{aligned}$$

where, in the last integral, I have assumed $0 < r < t$ and used the fact that $H(x;r)$ is an even function of r, so that the integrand is an odd function, and the integral over $[r-t, t-r]$ is zero. I have also used the corresponding evenness of G to replace the argument $r-t$ by $t-r$ in first two terms. It is now a simple matter to let $r \searrow 0$ to get

$$u(x,t) = \lim_{r\to 0} U(x;r,t) = G(x;t) + tG_r(x;t) + tH(x;t).$$

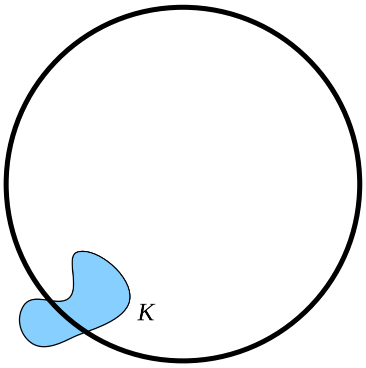
It only remains to compute G_r:

$$\frac{\partial}{\partial r}G(x;r) = \frac{\partial}{\partial r}\fint_{B(0,1)} g(x+rz)\,\mathrm{d}S(z) = \fint_{\partial B(0,1)} Dg(x+rz)\cdot z\,\mathrm{d}S(z)$$

which we translate back to an integral over $B(x,r)$ (remembering that $y = x + rz$ implies $z = (y-x)/r$) and we end with

$$u(x,t) = \fint_{\partial B(x,t)} \big(g(y) + Dg(y)\cdot(y-x) + th(y)\big)\,\mathrm{d}y$$

which is Kirchoff's formula.



#7.2: We use Kirchoff's formula (see above). Now that formula averages something which appears to grow like t, which doesn't look very promising. However, we have not yet used the compactness of the supports of g and h. This will imply that, when t is large, the joint support K of g and h will only intersect a small portion of $\partial B(x,t)$. More precisely, the surface area of $K \cap \partial B(x,t)$ will be bounded by some number, say A, while the total surface area is proportional to t^2. This should lead to the desired estimate:

$$\begin{aligned}
u(x,t) &= \frac{1}{4\pi t^2}\int_{K\cap\partial B(x,t)} \big(g(y) + Dg(y)\cdot(y-x) + th(y)\big)\,\mathrm{d}y \\
&\le \frac{A}{4\pi t^2}(\max|g| + t\max|Dg| + t\max|h|) \le \frac{C}{t}
\end{aligned}$$

as was to be proved.

#**7.3:** Accepting the statement of the hint without further proof (it seems rather obvious from a drawing, and is perhaps best proven by an application of the Hahn–Banach theorem), we integrate the inequality $\varphi(u(x)) \geq \varphi(\bar{u}) + a \cdot (u(x) - \bar{u})$ over A to obtain

$$\begin{aligned}
\fint_A \varphi(u(x))\,\mathrm{d}x &\geq \fint_A \left(\varphi(\bar{u}) + a \cdot (u(x) - \bar{u})\right)\mathrm{d}x \\
&= \varphi(\bar{u}) + a\Big(\fint_A u(x)\,\mathrm{d}x - \bar{u}\Big) \\
&= \varphi(\bar{u}) = \varphi\Big(\fint_A u(x)\,\mathrm{d}x\Big)
\end{aligned}$$

and we have the desired inequality. This proof only uses the linearity of $\fint_A$, plus its monotonicity ($\fint_A \varphi\,\mathrm{d}x \geq \fint_A \psi\,\mathrm{d}x$ if $\varphi \geq \psi$ on A) and the fact that $\fint_A 1\,\mathrm{d}x = 1$. These properties hold equally well for any kind of integral average, so Jensen's inequality holds for these. (The most obvious generalization, for those who know enough measure and integration theory, is obtained by integrating with respect to some probability measure.)

#**7.4:** In case you have not understood it yet, the previous exercise (which has little connection with PDEs) was there only so you could do this one. So let u be harmonic in Ω and φ be convex. If $B(x,r) \subset \Omega$ then by the mean value property of u and Jensen's inequality:

$$\varphi(u(x)) = \varphi\Big(\fint_{\partial B(x,r)} u\,\mathrm{d}S\Big) \leq \fint_{\partial B(x,r)} \varphi(u(y))\,\mathrm{d}S$$

which precisely states that $\varphi \circ u$ is subharmonic.

Problem set #8 — for next week

#**8.1:** Suppose $\Omega \subset \mathbb{R}^n$ is a bounded region with smooth boundary and $u \in C^2(\bar{\Omega})$ is harmonic in Ω. Show that

$$\int_{\partial\Omega} u\frac{\partial u}{\partial \nu}\,\mathrm{d}S \geq 0, \quad \int_{\partial\Omega} \frac{\partial u}{\partial \nu}\,\mathrm{d}S = 0.$$

#**8.2:** Suppose $\Omega \subset \mathbb{R}^n$ is a bounded region with smooth boundary. Study the uniqueness of solutions to the Neumann problem

$$\Delta u = 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial \nu} = g \text{ on } \partial\Omega,$$

if a solution exists. Explain why no solution can exist unless the compatibility condition $\int_{\partial\Omega} g\,\mathrm{d}S = 0$ is satisfied.

#**8.3:** Suppose $\Omega \subset \mathbb{R}^n$ is a bounded region. Is the solution (if it exists) for the problem $\Delta u = u^3$ in Ω, $u = g$ on $\partial\Omega$ unique?

#**8.4:** Evans problem 1 (p. 152).

#**8.5:** Evans problem 3 (p. 152).

#**8.6:** Evans problem 4 (p. 152).