75042 Partial differential equations

Problem set #8 — suggested solutions

#**8.1:** Aaarggh! Misprint! You were supposed to show

$$\int_{\partial\Omega} u\frac{\partial u}{\partial \nu}\,\mathrm{d}S \geq 0, \quad \int_{\partial\Omega} \frac{\partial u}{\partial \nu}\,\mathrm{d}S = 0$$

(these are *boundary* integrals) where u is harmonic and suitably smooth at the boundary.

The second one is almost trivial. This is just the divergence theorem:

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu}\,\mathrm{d}S = \int_{\partial\Omega} \nu\cdot Du\,\mathrm{d}S = \int_\Omega \operatorname{div} Du\,\mathrm{d}x = \int_\Omega \Delta u\,\mathrm{d}x = 0$$

The first one is just partial integration. I remind you that this is nothing but the divergence theorem applied to a product: Since $\operatorname{div}(uDu) = Du\cdot Du + u\operatorname{div} Du = |Du|^2 + u\Delta u = |Du|^2$ (when u is harmonic) we find

$$\int_{\partial\Omega} u\frac{\partial u}{\partial \nu}\,\mathrm{d}S = \int_{\partial\Omega} \nu\cdot uDu\,\mathrm{d}S = \int_\Omega \operatorname{div}(uDu)\,\mathrm{d}x = \int_\Omega |Du|^2\,\mathrm{d}x \geq 0$$

and, by the way, the inequality must be strict unless u is constant (on each component of Ω).

#**8.2:** We use the linearity. If u, v solve the given problem then their difference $w = u - v$ will solve

$$\Delta w = 0 \text{ in } \Omega, \quad \frac{\partial w}{\partial \nu} = 0 \text{ on } \partial\Omega.$$

Clearly, any constant function satisfies this; so the best we can ever hope to prove is that two solutions of the original problem differ by a constant. And we can indeed prove this, for if w is not constant then by the final remark in the solution of #8.1 above, we have

$$\int_{\partial\Omega} w\frac{\partial w}{\partial \nu}\,\mathrm{d}S > 0$$

which is clearly impossible since $\partial w/\partial\nu = 0$ on $\partial\Omega$. (One sometimes comes by mixed boundary value problems, in which u is given on part of the boundary and $\partial w/\partial\nu$ on the rest. Clearly the same proof works there, and in this case we do get uniqueness, provided only that u is given on a nonempty subset of the boundary.)

The compatibility condition $\int_{\partial\Omega} g\,\mathrm{d}S = 0$ is necessary for the existence of a solution because of the second formula of #8.1.

#**8.3:** The problem cannot have more than one solution. For if u and v solve the problem, let $w = u - v$. Then $w = 0$ on $\partial\Omega$. If w is not identically zero, then either $w > 0$ somewhere or $w < 0$ somewhere (or both). In the former case, w has a positive maximum at some interior point of Ω. At this point x, then, $\Delta w(x) = \Delta u(x) - \Delta v(x) = u(x)^3 - v(x)^3 > 0$ because $w(x) > 0 \Leftrightarrow u(x) > v(x)$. But elementary calculus tells us that we cannot have $\Delta w(x) > 0$ at a maximum point; no fancy PDE theory is required at all. (The argument works equally well if we replace u^3 by $f(u)$ where f is any strictly increasing function.)

#**8.4:** Clearly $u(x,t,a,b) = a\cdot x - tH(a) + b$ gives us $u_t = H(a)$ and $Du = a$ (the D operator here is assumed to only act on the x variables), so u is an integral of the Hamilton-Jacobi equation $u_t - H(Du) = 0$.

To show it is a *complete* integral, we must show the $(n+2)\times(n+1)$ matrix

$$\begin{bmatrix} u_{a_1} & \dots & u_{a_n} & u_b \\ u_{x_1a_1} & \dots & u_{x_1a_n} & u_{x_1b} \\ \vdots & \ddots & \vdots & \vdots \\ u_{x_na_1} & \dots & u_{x_na_n} & u_{x_nb} \\ u_{ta_1} & \dots & u_{ta_n} & u_{tb} \end{bmatrix} = \begin{bmatrix} x_1 - tH_{p_1}(a) & \dots & x_n - tH_{p_n}(a) & 1 \\ 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \\ H_{p_1} & \dots & H_{p_n}(a) & 0 \end{bmatrix}$$

has rank $n+1$. (This matrix is transposed relative to what the lecture note does, by the way.) The block consisting of rows $2,\dots,n+1$ and columns $1,\dots,n$ is an identity matrix. Clearly the rows $1,\dots,n+1$ are linearly independent, so the matrix has the necessary rank.

#**8.5:** I rename (x_1,x_2) to (x,y) for convenience.

(a) We solve $u_x+u_y=u^2$ with initial data $u(x,0)=g(x)$. The characteristic equations are

$$\dot{x}=1,\quad \dot{y}=1,\quad \dot{z}=z^2$$

and their solution is $x=t$, $y=t+A$, $z=1/(B-t)$. (There should have been an integration constant in the solution for x, but I threw it away because I am free to choose the starting point of a parametrization of the characteristics. Of course, I can only throw away one integration constant this way.) Eliminate t to describe the characteristic curves as

$$y=x+A,\quad z=\frac{1}{B-x}.$$

A characteristic lying on the graph of the solution u must fit the initial data $u(x,0)=g(x)$, so that $z=g(x)$ when (x,y,z) is on the characteristic with $y=0$. This happens when $x=-A$, so we must have

$$\frac{1}{B-(-A)}=g(-A) \text{ and therefore } B=-A+\frac{1}{g(-A)}.$$

Thus, passing back to the general form for the characteristic curve we get $A=y-x$, $B=x-y+1/g(x-y)$, and so

$$u(x,y)=z=\frac{1}{B-x}=\frac{g(x-y)}{1-yg(x-y)}$$

which does indeed solve the problem.

(b) We solve $xu_y-yu_x=u$ with initial data $u(x,0)=g(x)$. The characteristic equations are

$$\dot{x}=-y,\quad \dot{y}=x,\quad \dot{z}=z$$

with solutions $x=A\cos t$, $y=A\sin t$, $z=Be^t$. Here again we have thrown away a useless integration constant — to be added to t — in the expressions for x and y.

We note that the projected characteristics are circles in the (x,y) plane, and each of these cross the initial curve $y=0$ twice. Since $|z|$ grows as we travel along a characteristic with increasing t, the initial data cannot be satisfied everywhere. If, however, we restrict our attention to the right half plane $x>0$ there is no trouble at all.

We eliminate t from our equations for the characteristic and get $x^2+y^2=A^2$ and $z=B\exp\big(\arctan(y/x)\big)$. The initial data implies that we should have $z=g(x)$ when $y=0$. At this point on the characteristic $x=A$, so we get Thus $B=g(A)$. In general, then, our solution becomes

$$u(x,y)=z=g\left(\sqrt{x^2+y^2}\right)\exp\left(\arctan\frac{y}{x}\right).$$

#**8.6:** *Let it wait until next week.* This problem set was too big, perhaps. To compensate, I'll make the next one too small.

Problem set #9 — for next week

#**9.1:** Evans problem 5 (p. 152).

#**9.2:** Evans problem 6 (p. 152).