75042 Partial differential equations

Problem set #10 — suggested solutions

(I am doing #10.2 first to solve a slight layout problem.)

#10.2: This is a conservation law with flux function $F(u) = u(1-u) = u - u^2$ and characteristic speed $F'(u) = 1-2u$. The flux function is concave, so the solution to a Riemann problem with left state u_l and right state u_r is a expansion wave if $u_l > u_r$ (since then $F'(u)$ increases as u moves from u_l towards u_r) and a shock wave if $u_r > u_l$.

The given initial value problem is just like two Riemann problems, at least for small t. There is one at the discontinuity at $x = 0$, and one at $x = 1$ (a Riemann problem with an initial discontinuity at $x = 1$ is completely equivalent to the corresponding problem with the discontinuity at $x = 0$, since replacing u by $u(x-1,t)$ gives us a new solution of the conservation law but with the initial discontinuity moved.)

We get to consider two cases:

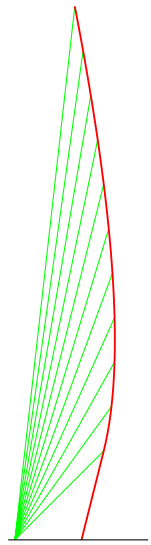
- $u_1 < u_0$ — expansion wave at $x = 0$, shock at $x = 1$;
- $u_1 > u_0$ — shock at $x = 0$, expansion wave at $x = 1$.

in each case the shock speed is

$$\sigma = \frac{[u - u^2]}{[u]} = \frac{u_1 - u_1^2 - u_0 + u_0^2}{u_1 - u_0} = 1 - u_1 - u_0$$

which is the mean value of the two characteristic speeds $F'(u_\pm)$ (as is always the case with a quadratic flux function). In either case, therefore, the shock will sooner or later run into the expansion wave. The shock will from then on have a constant state on one side and a variable state on the other side, and so the shock speed will change.

We consider in detail only the first case ($u_1 < u_0$). The solution is shown in the margin (for $u_1 = 0$ and $u_0 = 3/4$). Only part of the expansion wave is shown; to the righthand side of the shock, the characteristic speed is $-1/2$.



In the expansion wave, $F'(u) = 1 - 2u = x/t$ so that $u = \frac{1}{2}(1 - x/t)$. The righthand edge of the expansion wave is the characteristic $x = (1-2u_1)t$. The shock has speed $1 - u_0 - u_1$ and therefore equation $x = 1 + (1 - u_0 - u_1)t$. The two lines meet at (x_0, t_0):

$$x_0 = \frac{1-2u_1}{u_0 - u_1}, \quad t_0 = \frac{1}{u_0 - u_1},$$

so that point marks the beginning of the curved part of the shock, which must satisfy the equation (the Rankine–Hugoniot relation)

$$\dot{s} = 1 - u_0 - \frac{1}{2}\left(1 - \frac{s}{t}\right) = \frac{1}{2} - u_0 + \frac{s}{2t}.$$

This equation is easily solved: Solve the corresponding homogeneous equation $\dot{s} = s/(2t)$ first to find $s = C\sqrt{t}$, and then try $s(t) = q(t)\sqrt{t}$ etc. I admit I got lazy and used Maple to solve the equation together with the initial data $s(t_0) = x_0$, and found the solution

$$s = (1-2u_0)t + 2\sqrt{(u_0 - u_1)t}.$$

Note that $\lim_{t\to\infty} \dot{s}(t) = 1 - 2u_0$ which is the characteristic speed on the right side of the shock. Hence the characteristic speed on the left side has the same limit (the shock speed being the average of the two characteristic speeds), and the value of u on the left side approaches u_0. This means that the strength of the shock approaches zero as $t \to \infty$. As the left edge of the expansion wave is at $x = (1-2u_0)t$ the width of the wave is $2\sqrt{(u_0 - u_1)t}$ which grows without bound as $t \to \infty$. For large t, therefore, the solution is a very wide wave with values near the constant u_0 and a weak shock at the right side.

The other case ($u1 > u_0$) is similar, and can in fact be reduced to the former case by considering $w(x,t) = 1 - u(1-x,t)$.

#**10.1:** Write a "small" wave as $u(x,t) = u_0 + v(x,t)$ where v is small. The equation $u_t + f(u)_x = 0$ in quasilinear form $u_t + f'(u)u_x = 0$ is then equivalent to $v_t + f'(u_0 + v)u_x = 0$, and it seems reasonable to replace $f'(u_0+v)$ by $a = f'(u_0)$. Thus we are led to the transport equation $v_t + av_x = 0$, whose general solution is $v = g(x - at)$ where g is an arbitrary function. "Small" waves, therefore, travel with a constant speed $f'(u_0)$ corresponding to the sound speed in acoustics.

The above discussion may be made more complete by considering discontinuous functions v. A shock in v, as long as $|v|$ itself is small, is just a small shock in u. If the left and right states $u_\pm$ are both close to u_0 the Rankine-Hugoniot condition becomes

$$\sigma = \frac{[f(u)]}{[u]} = \frac{f(u_+) - f(u_-)}{u_+ - u_-} = f'(\tilde{u}) \approx f'(u_0)$$

where $\tilde{u}$ is between u_+ and u_-. Thus shocks, too, move with the approximate speed of $f'(u_0)$.

#**10.3:** [Exam 1995–08–30] This problem is a conservation law with flux function $F(u) = u^3/3$ and characteristic speed $F'(u) = u^2$. The characteristic starting at $(x,t) = (\xi, 0)$ will therefore have speed $c(\xi) = u(\xi,0)^2 = \xi^2$, and its equation is $x = \xi + \xi^2 t$. To find which characteristic passes through the point (x,t) we solve this equation for ξ and find two solutions for every (x,t) with $xt > -1/4$:

$$\xi = \frac{-1 \pm \sqrt{1 + 4xt}}{2t}.$$

On the other hand, when $xt < -1/4$ there is no solution, and this happens for arbitrarily small values of t. The deeper reason for the trouble becomes obvious if you make a drawing of the characteristics: It is the fact that for large negative ξ, the corresponding characteristic has a large positive speed. It is the unboundedness of characteristic speeds that causes the trouble.

I'll delay the solution of the (b) point until next week.

Problem set #11 — for next week

#11.1: [Exam 1995–12–22] Let v be a C^1 function of two variables (x,t) and let $u(x,t) = e^{v(x,t)}$. Show that u satisfies

$$u_t + (u^2)_x = 0$$

if and only if

$$v_t + 2(e^v)_x = 0,$$

but that the two equations are not equivalent if v is only assumed piecewise smooth and we interpret the problem in the weak sense.

#11.2: [Exam 1995–12–22] Find weak entropy solutions (i.e., weak solutions satisfying the entropy condition) for the Riemann problem

$$u_t + \frac{1}{4}\big((u^2-\pi^2)^2\big)_x = 0 \qquad (x \in \mathbb{R}, t > 0) \tag{1}$$

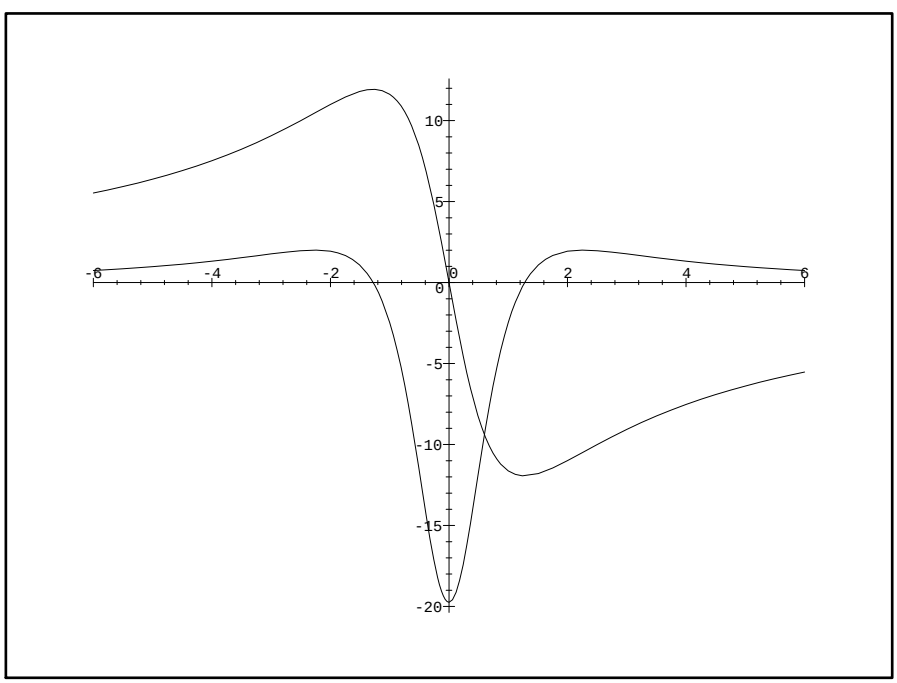
$$u(x,0) = \begin{cases} u_l & x < 0 \\ u_r & x > 0 \end{cases} \tag{2}$$

in each of the two cases

(i) $u_l = -\pi, \quad u_r = \pi$
(ii) $u_l = \pi, \quad u_r = -\pi$

(In one of the cases you would need to solve a third degree equation to wrote the answer in explicit form. Don't do this, but leave the equation in implicit form.)

Consider the initial value problem for (1) in the two cases

(iii) $u(x,0) = 2\operatorname{arctg} x$
(iv) $u(x,0) = -2\operatorname{arctg} x$

Why would you expect the solutions to have certain characteristics in common with the solutions from the previous question? Draw diagrams showing characteristics and shocks in the (x,t) plane in the two cases, and in particular indicate the times and positions where shocks first arise. You may, without further calculation, use the information you can extract from the graph of the function $c(\xi) = 8\operatorname{arctg}^3 \xi - 2\pi^2 \operatorname{arctg} \xi$ and its derivative in the figure below.