75042 Partial differential equations

Problem set #11 — suggested solutions

#10.2(b): The initial data has characteristic speeds $c(\xi) = F'(u(x,0))$ which is 1 when $|\xi| > 1$ and ξ^2 when $|\xi| \le 1$. Concentrating on the latter case, we write the equation of the characteristic starting at $(\xi, 0)$ as $x = \xi + \xi^2 t$. We expect a shock to appear when $x_\xi = 1 + 2\xi t$ becomes zero. This happens at $t = -1/2\xi$. The smallest such t is when the shock first appears: This is at $\xi = -1$ which gives $t = 1/2$ and $x = -1/2$. We solve the equation $x = \xi + \xi^2 t$ for ξ and find $\xi = (-1 \pm \sqrt{1+4xt})/(2t)$. We should pick the plus sign (with the minus sign, ξ becomes large and negative for small t). Thus

$$\xi = \frac{\sqrt{1+4xt}-1}{2t} = \frac{2x}{1+\sqrt{1+4xt}}$$

We end up with

$$u(x,t) = \begin{cases} -1 & x < t-1 \\ \frac{4x^2}{\left(1+\sqrt{1+4xt}\right)^2} & t-1 \le x \le t+1 \\ 1 & x > t+1 \end{cases}$$

when $0 \le t \le 1/2$. The shock that started at $x = -1/2$, $t = 1/2$ will have $u = -1$ on its lefthand side and u given by the fraction above on the righthand side. Initially the shock is very weak and has speed 1, but then it slows down as the righthand value begins to rise. In the limit as $t \to \infty$ the righthand value would approach the value v given by $F'(v) = (F(v) - F(-1))/(v+1)$, for at this righthand value of u the shock speed equals the righthand characteristic speed, and the shock will not overtake any more characteristics. That equation is simplified to $2v^3 + 3v^2 - 1 = 0$. It can be solved by noting that $v = -1$ is a solution, so we divide: $2v^3 + 3v^2 - 1 = (v+1)(2v^2 + v - 1)$. We are not interested in the solution $v = -1$ so we put $2v^2 + v - 1 = 0$ and get $v = 1/2$ (the other solution is $v = -1$ again). All in all, then, for $t > 1/2$ the solution ought to be given by

$$u(x,t) = \begin{cases} -1 & x < s(t) \\ \frac{4x^2}{\left(1+\sqrt{1+4xt}\right)^2} & s(t) \le x \le t+1 \\ 1 & x > t+1 \end{cases}$$

where the shock position s can be found from an ordinary differential equation (the Rankine–Hugoniot relation). For large t I expect $s(t) \approx t/4 + o(t)$.

#11.1: That the two equations $u_t + (u^2)_x = 0$ and $v_t + 2(e^v)_x = 0$ are equivalent when v is smooth and $u(x,t) = e^{v(x,t)}$ is a trivial exercise (get $u_t + (u^2)_x = (e^v)_t + (e^{2v})_x = e^v v_t + 2e^{2v} v_x = e^v(v_t + 2e^v v_x) = e^v(v_t + (e^v)v_x)$.)

Consider now a shock in v with left and right states $v_\pm$ and let $u_\pm = e^{v_\pm}$. The Rankine–Hugoniot shock condition for the v shock would be $e^{v+} - e^{v-} = \sigma(v_+ - v_-)$ while the corresponding shock condition for the u shock is $(u_+^2 - u_-^2) = \sigma(u_+ - u_-)$ which is equivalent to $u_+ + u_- = \sigma$, and which we then rewrite as $e^{v+} - e^{v-} = \sigma$. Clearly the two shock conditions are not equivalent, except in the exceptional case $v_+ - v_- = 1$.

#11.2: From the graph of the flux function $F(u) = (u^2 - \pi^2)^2/4$ and the rubber band method we immediately see that the solution in case (i) is a shock with speed 0, so $u(x,t) = -\pi$ when $x < 0$ and $u(x,t) = \pi$ when $x > 0$.

In case (ii) the rubber band method yields a shock from π to some value v, then an expansion wave from v to $-v$, and finally a shock from $-v$ to

$-\pi$. The expansion wave will have the implicit equation $F'(u) = x/t$, that is, $(u^2-\pi^2)u = x/t$ (we don't try to solve this), while v can be found from the requirement that the secant of the graph from v to π should have the same slope as the graph of F at v, that is $\frac{1}{4}(v^2-\pi^2)^2/(v-\pi) = (v^2-\pi^2)v$. This equation can actually be solved to get $v = \pi/3$. The corresponding speed is $(v^2 - \pi^2)v = -8\pi/27$. So the expansion wave is in the region $|x| < 8\pi t/27$ while $u = \pm\pi$ outside.

In cases (iii) and (iv) we expect the solutions, as "seen from a distance", to be like the solutions of (i) and (ii). For if we rescale the problem, replacing (x,t) by (ax,at), the conservation law is unchaged while the initial data is subject to the x scaling. So the function $u(ax,at)$ solving (iii) or (iv) should look like the solution of (i) or (ii) when a becomes large, because $\pm 2\operatorname{arctg}(ax)$ looks like the initial data for (i) or (ii) as a becomes large.

The detailed solution will have to wait until next week.

Problem set #12 — for next week (last week of the term!)

#**12.1:** [Exam 1995–08–30] Decide where the *Tricomi equation*

$$u_{xx} + xu_{yy} = 0$$

is elliptic, where it is parabolic, and where it is hyperbolic. Find the characteristic curves in the hyperbolic area.