75042 Partielle differensialligninger

Oppgavesett #1 (til 1997–09–02) — løsningsforslag

Oppgaven denne uken er tatt fra notatet *Elements of analysis* (EA). Løsningsforslaget er på engelsk av gjenbruksgrunner (av og til har vi utenlandske studenter som tar kurset).

EA–2: To show that $B_\varepsilon(x)$ is open, let $y \in B_\varepsilon(x)$. Let $\delta = \varepsilon - d(x,y)$. Then $\delta > 0$, and $B_\delta(y) \subseteq B_\varepsilon(x)$, since $d(y,z) < \delta \Rightarrow d(x,z) \le d(x,y) + d(y,z) < d(x,y) + \delta = \varepsilon$.

Next, let A be a set and A° its interior. If $x \in A^\circ$ then by definition there is some $\varepsilon > 0$ so $B_\varepsilon(x) \subseteq A$. However if $y \in B_\varepsilon(x)$ then some δ–neighbourhood of y is contained in $B_\varepsilon(x)$ and therefore in A. Thus A is a neighbourhood of y. In other words, $y \in A^\circ$. This shows that A° is open.

Finally, if $y \in X \setminus \bar{B}_\varepsilon(x)$ let $\delta = d(x,y) - \varepsilon$. Then $\delta > 0$ and $B_\delta(y) \subseteq X \setminus \bar{B}_\varepsilon(x)$, for if $d(y,z) < \delta$ and $z \in \bar{B}_\varepsilon(x)$ then $d(x,y) \le d(x,z) + d(z,y) < \varepsilon + \delta = d(x,y)$ which is impossible.

EA–3: Assume the sequence (x_n) has the two limits y and z. By the triangle inequality $d(y,z) \le d(y,x_n) + d(x_n,z)$ for any n. Given any $\varepsilon > 0$ we can make sure that $d(x_n,y) < \varepsilon$ and $d(x_n,z) < \varepsilon$ for n large enough. If we pick $\varepsilon = d(y,z)/2$ a contradiction results.

EA–4: Assume the subsequence (y_k) given by $y_k = x_{n_k}$ has the limit x. For any given $\varepsilon > 0$, there exists some K so that $d(y_k,x) < \varepsilon$ whenever $k \ge K$. Given any N, we find some K' so that $n_k \ge N$ whenever $k \ge K'$. Whenever $k \ge \max(K,K')$ then $n_k \ge N$ and $d(x_{n_k},x) < \varepsilon$. This shows that x is a limit point for the sequence (x_n).

On the other hand, if x is a limit point for (x_n) then pick any sequence of real numbers $\varepsilon_k > 0$ with $\varepsilon_k \searrow 0$ (by this we mean that the sequence (ε_k) is nonincreasing and converges to 0). Let n_1 be so that $d(x_{n_1},x) < \varepsilon_1$, and for $k = 2,3,\ldots$ let $n_k > n_{k-1}$ be so that $d(x_{n_k},x) < \varepsilon_k$ (the existence of these is exactly the definition of limit point). Now $x_{n_k} \to x$, for if $\varepsilon > 0$, we find first some K with $\varepsilon_K < \varepsilon$; and then, whenever $k \ge K$, we have $d(x_{n_k},x) < \varepsilon_k \le \varepsilon_K < \varepsilon$.

EA–5: We prove $(a) \Rightarrow (d) \Rightarrow (c) \Rightarrow (b) \Rightarrow (a)$.

$(a) \Rightarrow (d)$: Assume F is closed. Given a sequence in F with a limit point x, that sequence has a subsequence converging to x. Then $x \in F$ because F is closed.

$(d) \Rightarrow (c)$: Assume F contains every limit point of every sequence in F. Assume $x \in X$ is such that every neighbourhood of x meets F (we say A *meets* B if $A \cap B \ne \emptyset$). Let $\varepsilon_n > 0$ with $\varepsilon_n \searrow 0$. For each n, let $x_n \in F \cap B_{\varepsilon_n}(x)$. Then $x_n \to x$, and so $x \in F$ because x is a limit point of the sequence (x_n).

$(c) \Rightarrow (b)$: Let $x \in X \setminus F$. If (c) holds, then there must exist a neighbourhood of x which does not meet F. This neighbourhood, then, is contained in $X \setminus F$ which is itself, therefore, a neighbourhood of x. Thus $X \setminus F$ is a neighbourhood of all its points, and is therefore open.

$(b) \Rightarrow (a)$: Assume $X \setminus F$ is open and let (x_n) be a sequence in F, $x_n \to x$. If $x \notin F$ then $x \in X \setminus F$, so $X \setminus F$ is a neighbourhood of x. This neighbourhood must then contain a tail of the sequence (x_n), which is impossible since $x_n \in F$ for all n.

EA–12: For all n, we have $\inf\{x_k : k \ge n\} \le \sup\{x_k : k \ge n\}$. Taking limits in this inequality immediately yields $\underline{\lim}\, x_n \le \overline{\lim}\, x_n$.

Assume $L = \overline{\lim}\, x_n$ is finite. Since $S_n = \sup\{x_k : k \ge n\} \searrow L$, then given $\varepsilon > 0$ there is some N so that $L \le S_n < L + \varepsilon$ whenever $n \ge N$. By the definition of the supremum there is some $k \ge n$ so that $S_n - \varepsilon < x_k \le S_n$. For any such k, then $|x_k - L| \le |x_k - S_n| + |S_n - L| < \varepsilon + \varepsilon = 2\varepsilon$. Thus L is a limit point for (x_n).

No limit point x exists with $x > \overline{\lim}\, x_n$, for if this inequality holds then for some sufficiently large n, $\sup\{x_k : k \ge n\} < x$. Then this tail of the sequence does not meet $B_\varepsilon(x)$ where $\varepsilon = x - \sup\{x_k : k \ge n\}$, so x is not a limit point.

Similarly to the above two results $\underline{\lim}\, x_n$ is a limit point, and no limit point x with $x < \underline{\lim}\, x_n$ can exist. This is most easily seen from the previous results and the identity $\underline{\lim}\, x_n = -\overline{\lim}(-x_n)$ (or you can go through the proofs, reversing inequalities).

If $x_n \to x \in \mathbb{R}$ and $\varepsilon > 0$, we get in the usual way $x - \varepsilon < x_n < x + \varepsilon$ whenever $n \ge N$ — for some N. Then $x - \varepsilon \le \inf\{x_n : n \ge N\} \le \sup\{x_n : n \ge N\} \le x + \varepsilon$. Taking limits, we get $x - \varepsilon \le \underline{\lim}\, x_n \le \overline{\lim}\, x_n \le x + \varepsilon$. Since this holds for any $\varepsilon > 0$, we must have $\underline{\lim}\, x_n = \overline{\lim}\, x_n = x$.

On the other hand, if $\underline{\lim}\, x_n = \overline{\lim}\, x_n = x \in \mathbb{R}$, then whenever $\varepsilon > 0$ we have $x - \varepsilon < \inf\{x_n : n \geq N\} \leq \sup\{x_n : n \geq N\} < x + \varepsilon$ for sufficiently large N, and so whenever $n \geq N$, $x - \varepsilon < x_n < x + \varepsilon$. In other words $|x_n - x| < \varepsilon$, so $x_n \to x$.

EA–13: It seems to me that the only unproved points in the proof are that Cauchy sequences are bounded (but that is too trivial to bother with here?) and that any limit point of a Cauchy sequence is in fact the limit of that sequence. So to prove the latter, let X be any metric space (complete or not), and let (x_n) be a Cauchy sequence in X with a limit point x. To show that $x_n \to x$, let $\varepsilon > 0$. Since (x_n) is a Cauchy sequence, there is some N so that $d(x_m, x_n) < \varepsilon$ whenever $m, n \geq N$. But since x is a limit point, there is at least one point x_m with $m \geq N$ so that $d(x_m, x) < \varepsilon$. But then, given *any* $n \geq N$ we have $d(x_n, x) \leq d(x_n, x_m) + d(x_m, x) < \varepsilon + \varepsilon = 2\varepsilon$. And so we have proved $x_n \to x$.

EA–15: Let K be a closed and bounded subset of $\mathbb{R}$ and let (x_n) be a sequence in K. Since K is bounded, so is (x_n), and by Theorem 14, that sequence has at least one limit point in $\mathbb{R}$. Since K is closed, that limit point is in K. This proves that K is compact.

EA–16: Let X be a compact metric space and K a closed subset of X. If (x_n) is a sequence in K then it has a limit point in X, and since K is closed, that limit point belongs to K.

Alternative proof: Let $\mathscr{V}$ be an open cover of K in the following sense: Each member of $\mathscr{V}$ is an open subset of X, and their union contains K. Then $\mathscr{V} \cup \{X \setminus K\}$ is an open cover of X, and so has a finite subcover $\mathscr{U}$ since X is compact. This will also be a finite cover of K. This proves that K is compact provided the following is clear:

> Let X be a metric space and Y any subset of X. Then, as remarked in the notes, Y is itself a metric space. An open subset of Y is defined in the usual way. It turns out that a subset $V \subseteq Y$ is open in Y if and only if we can write $V = Y \cap U$ where $U \subseteq X$ is open in X. In fact, for each $v \in V$ we can find $\varepsilon(v) > 0$ so that $y \in V$ whenever $y \in Y$ and $d(y, v) < \varepsilon(v)$. Let $U = \bigcup_{v \in V} B_{\varepsilon(v)}(v)$ (a union of balls in X). Then U is open, and $Y \cap U = V$.

EA–23: Consider the triangle inequality $d(x, a) \leq d(x, y) + d(y, a)$. With $a \in A$ this implies $\operatorname{dist}(x, A) \leq d(x, y) + d(y, a)$. Since this holds for all a, we can take the infimum to get $\operatorname{dist}(x, A) \leq d(x, y) + \operatorname{dist}(y, A)$. In other words we have $\operatorname{dist}(x, A) - \operatorname{dist}(y, A) \leq d(x, y)$. The same inequality with x and y interchanged yields $|\operatorname{dist}(x, A) - \operatorname{dist}(y, A)| \leq d(x, y)$, which clearly shows that $\operatorname{dist}(x, A)$ is a continuous function of x.

If $K \cap \bar{A} = \emptyset$, then the continuous function $x \mapsto \operatorname{dist}(x, A)$ is positive on K, for $\operatorname{dist}(x, A) > 0$ if and only if $x \notin \bar{A}$. If K is compact, that function achieves its minimum on K, and so there is a constant $\varepsilon > 0$ so that $\operatorname{dist}(x, A) \geq \varepsilon$ for each $x \in K$.

Oppgavesett #2 (til 1997–09–16) — til neste gang

Fritz John side 18–19: 1, 2, 6. Dessuten:

#2–1: Klassifisér følgende ligning og bestem dens karakteristikker:

$$(x^2 - 1)u_{xx} + (y^2 - 1)u_{yy} + 2xyu_{xy} = 0.$$