75042 Partielle differensialligninger

Oppgavesett #8 (til 1997–10–28) — løsningsforslag

Evans 2.5–9: Extend u to a function v on the full ball by antisymmetry in the x_n variable, as explained in the problem. The function

$$w(x) = \int_U K(x,y)v(y)\,\mathrm{d}S(y)$$

is then continuous in $\overline{U}$ (v is certainly continuous on ∂U, since $u = 0$ on $x_n = 0$) and harmonic in U. The Poisson kernel $K(x,y)$ is clearly symmetric in y_n when $x_n = 0$ (see its formula!) and therefore, when $x_n = 0$ the integrand above is antisymmetric in y_n. Thus $w(x) = 0$ when $x_n = 0$. Since w is harmonic in U^+ with the same boundary values as u, $w = u$ in all of U^+. Further, $w(x)$ is antisymmetric in x_n and therefore $w = v$. This antisymmetry can be seen directly (note that $K(x,y)$ in unchanged if x_n and y_n both are replaced by $-x_n$ and $-y_n$ respectively).

Alternatively, one can note that the function $-w(x_1,\dots,x_{n-1},-x_n)$ is harmonic with the same boundary values as w to get the antisymmetry (and in particular the vanishing of $w(x)$ when $x_n = 0$).

Evans 2.5–11: That $u_\lambda(x,t) = u(\lambda x, \lambda^2 t)$ solves $u_t - \Delta u = 0$ if u does so, is trivial, since the chain rule yields $u_{\lambda,t} - \Delta u_\lambda = \lambda^2(u_t - \Delta u)$.

(In cases like this where not all subscripts mean differentiation, is is handy to introduce a comma to separate those subscripts that indicate a derivative from those that do not.)

Now differentiate u_λ with respect to λ. Equality of mixed partials, i.e., $u_{\lambda,\lambda t} = u_{\lambda,t\lambda}$ and $\Delta u_{\lambda,\lambda} = (\Delta u_\lambda)_\lambda$ guarantee that $u_{\lambda,\lambda}$ satisfies the heat equation. Put $\lambda = 1$ to obtain the solution $v = x\cdot \mathrm{D}u + 2tu_t$.

#8–1: An unusual but quick way to derive the Laplacian in polar coordinates, is the following: Apply the divergence theorem to $\mathrm{D}u$ in the region $r_1 < r < r_2$, $\theta_1 < \theta < \theta_2$:

$$\begin{aligned}\iint_R \Delta u\,\mathrm{d}x\mathrm{d}y &= \int_{r_1}^{r_2} \frac{-u_\theta(r,\theta_1)}{r}\,\mathrm{d}r + \int_{\theta_1}^{\theta_2} u_r(r_2,\theta)\,r_2\mathrm{d}\theta + \int_{r_1}^{r_2} \frac{u_\theta(r,\theta_2)}{r}\,\mathrm{d}r + \int_{\theta_1}^{\theta_2} -u_r(r_1,\theta)\,r_1\mathrm{d}\theta \\ &= \int_{r_1}^{r_2}\int_{\theta_1}^{\theta_2} \Big((ru_r)_r + \frac{u_{\theta\theta}}{r}\Big)\,\mathrm{d}\theta\,\mathrm{d}r\end{aligned}$$

which, when compared to the formula $\iint_R \Delta u\,\mathrm{d}x\mathrm{d}y = \int_{r_1}^{r_2}\int_{\theta_1}^{\theta_2} \Delta u(r,\theta)\,\mathrm{d}\theta\,r\mathrm{d}r$ yields the inevitable conclusion

$$\Delta u = \frac{(ru_r)_r}{r} + \frac{u_{\theta\theta}}{r^2}.$$

Attempting $u(r,\theta) = R(r)\Theta(\theta)$ we immediately arrive at $(rR')'\Theta/r + R\Theta''/r^2 = 0$ which we multiply by $r^2/(R\Theta)$ to get $r(rR')'/R + \Theta''/\Theta = 0$, so that $r(rR')'/R = -\Theta''/\Theta$ is constant. Since Θ must be periodic with period 2π this constant must be of the form n^2 with n an integer, and we may write $\Theta = e^{in\theta}$. The R equation may be written $r(rR')' = n^2 R$, which has the solutions (linear combinations of) $R = r^{\pm n}$. We want our solutions to be continuous at the origin, so from these two we pick only $R = r^{|n|}$. (For $n = 0$ the general solution is $R = A + B\ln r$, and again we must throw away the part that is singular at the origin, leaving $R = 1 = r^{|0|}$.)

Taking linear combinations of the product solutions, we are therefore led to trying a series

$$u(x,y) = \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{in\theta}$$

where $x = r\cos\theta$ and $y = r\sin\theta$ as usual. Clearly, each term in the series is harmonic. When we set $r = 1$ in the series and use the boundary conditions $u(\cos\theta,\sin\theta) = g(\theta)$, we simply get the Fourier series for g. Thus we should have

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\varphi)e^{-in\varphi}\,\mathrm{d}\varphi.$$

We insert this back into the series for u, and interchange the order of summation and integration to get

$$u(x,y) = \int_{-\pi}^{\pi} g(\varphi)K(r,\theta-\varphi)\,\mathrm{d}\varphi$$

where P is the famous *Poisson kernel*:

$$\begin{aligned} P(r,\theta) &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta} = \frac{1}{2\pi}\left(\sum_{n=0}^{\infty} r^n e^{in\theta} + \sum_{n=0}^{\infty} r^n e^{-in\theta} - 1\right) \\ &= \frac{1}{2\pi}\left(\frac{1}{1-re^{i\theta}} + \frac{1}{1-re^{-i\theta}} - 1\right) = \frac{1}{2\pi}\frac{1-r^2}{1+r^2-2r\cos\theta}. \end{aligned}$$

The general Poisson formula for $n=2$ (Evans page 43) is

$$u(x) = \int_{\partial B(0,1)} K(x,y)g(y)\,\mathrm{d}S(y), \qquad K(x,y) = \frac{1-|x|^2}{2\pi|x-y|^2} \quad (x,y\in\mathbb{R}^2).$$

In this formula, write $x = re^{i\varphi}$ and $y = e^{i(\varphi+\theta)}$ and note that then $|x-y|^2 = |r-e^{i\theta}|^2 = 1+r^2-2\cos\theta$, so this P is really K expressed in polar coordinates.

Oppgavesett #9 (til 1997–11–04)

Fra Evans side 82–83: Oppgave 13,14,16. (Det er en tyrkleif i oppgave 14: Du må legge til $h(t)$ i formelen for $u(x,t)$.)

#9–1: [Oppgaven er gjenbrukt fra i fjor, derfor på engelsk.] We say a continuous function u is *subharmonic* in an open set Ω if $u(x) \le ⨍_{\partial B(x,r)} u\,\mathrm{d}S$ whenever $B(x,r)\subset\Omega$. Note that we do not require u to be differentiable. Prove that, if u is subharmonic and v is harmonic in Ω and both functions are continuous on $\overline{\Omega}$ with $u\le v$ on $\partial\Omega$, then $u\le v$ in Ω. Also prove that, if $u\in C^2(\Omega)$, then u is subharmonic in the present sense if and only if $-\Delta u \le 0$ in Ω.