75042 Partielle differensialligninger

Oppgavesett #9 (til 1997–11–04) — løsningsforslag

Evans 2.5–13: With $v(x,t) = e^{ct}u(x,t)$ we see that v satisfies the equation $v_t - \Delta v = e^{ct}f$ with the initial value $v(x,0) = g(x)$. There is an explicit formula for the solution to this problem on the top of p. 53. Just replace u by v and $f(y,s)$ by $e^{cs}f(y,s)$ in that formula, and finally set $u(x,t) = e^{-ct}v(x,t)$. I omit the full detailed formula here.

Evans 2.5–14: Extend $v(x,t) = u(x,t) - h(t)$ to $x < 0$ by odd reflection: $v(-x,t) = -v(x,t)$. Clearly, v is continuous at $x = 0$ (because of the boundary condition $u(0,t) = h(t)$). When $x > 0$ we find $v_t - v_{xx} = u_t - u_{xx} - h'(t) = -h'(t)$. The same relation, with a sign reversal, will hold for $x < 0$ by antisymmetry, so we have

$$v_t - v_{xx} = \begin{cases} -h'(t) & x > 0, \\ h'(t) & x < 0. \end{cases}$$

Now, the solution to the problem $v_t - v_{xx} = f(x,t)$ with $v(x,0) = 0$ is given by Duhamel's principle (where we replace $f(y,s)$ by the righthand side of the above equation of course):

$$\begin{aligned} v(x,t) &= \int_0^t \frac{1}{\sqrt{4\pi(t-s)}} \int_{\mathbb{R}} \exp\left[-\frac{(x-y)^2}{4(t-s)}\right] f(y,s)\,\mathrm{d}y\,\mathrm{d}s \\ &= \int_0^t \frac{1}{\sqrt{4\pi(t-s)}} \left(\int_{-\infty}^0 - \int_0^\infty\right) \exp\left[-\frac{(x-y)^2}{4(t-s)}\right] \mathrm{d}y\, h'(s)\,\mathrm{d}s \\ &= \int_0^t \underbrace{\frac{1}{\sqrt{\pi}} \left(\int_{x/\sqrt{4(t-s)}}^\infty - \int_{-\infty}^{x/\sqrt{4(t-s)}}\right) \exp[-z^2]\,\mathrm{d}z}_{k(s)}\; h'(s)\,\mathrm{d}s \\ &= k(t)h(t) - k(0)h(0) - \int_0^t k'(s)h(s)\,\mathrm{d}s \end{aligned}$$

where we note that $k(t) = 0$, $h(0) = 0$, and $k'(s)$ is easily computed as

$$k'(s) = -\frac{x}{\sqrt{4\pi}(t-s)^{3/2}} \exp\left[-\frac{x^2}{4(t-s)}\right].$$

From this we obtain the required formula, which is as shown in the problem with the term $h(t)$ added (clearly there is a misprint).

Evans 2.5–16: It is probably a good idea to extend our given function $g \in C([0,1])$ to have compact support, say $g(x) = 0$ when $x \notin [-1,2]$, and let g vary linearly from 0 to $g(0)$ on the interval $[-1,0]$ and linearly from $g(1)$ to 0 on $[1,2]$. We let

$$u(x,t) = \int_{\mathbb{R}} \Phi(y,t)g(x-y)\,\mathrm{d}y = \int_{x-2}^{x+1} \Phi(y,t)g(x-y)\,\mathrm{d}y \qquad (1)$$

for $t > 0$, and $u(x,0) = g(x)$. Here Φ is the standard heat kernel. That u does in fact solve the heat equation is of no use to us here, but its continuity in $t \geq 0$ is useful: Not only does $u(x,t) \to g(x)$ as $t \searrow 0$, but u is uniformly continuous on every compact subset of the closed upper half plane $t \geq 0$, and therefore the function $x \mapsto u(x,t)$ converges uniformly to g on every bounded interval — and in particular it does so on $[0,1]$.

In other words, given $\varepsilon > 0$ we can find $\delta > 0$ so that $|u(x,t) - g(x)|$ whenever $0 \leq x \leq 1$ and $0 < t < \delta$. From now on, we fix one such t. It only remains to approximate $u(x,t)$ by a polynomial.

Now we use the explicit form of the heat kernel:

$$\Phi(y,t) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{y^2}{4t}\right)$$

The plan is to replace the exponential in Φ by a partial sum of the Taylor series $\exp(z) = \sum_{k=0}^{\infty} z^k/k!$. We are interested in $u(x,t)$ for $x \in [0,1]$. We see that the integral in (1) only uses values of $\Phi(y,t)$ with $|y| \le 3$, which implies $y^2/(4t) \le 9/(4t)$.

For our particular fixed t, let n be so large that $|e^z - \sum_{k=0}^{n} z^k/k!| < \varepsilon\sqrt{4\pi t}$ whenever $|z| \le 9/(4t)$ (just make sure that $\sum_{k=n+1}^{\infty} 9^k/(k!(4t)^k) < \varepsilon\sqrt{4\pi t}$). Let

$$\Psi(y,t) = \frac{1}{\sqrt{4\pi t}} \sum_{k=0}^{n} \frac{y^{2k}}{k!(4t)^k}$$

so that $|\Psi(y,t) - \Phi(y,t)| < \varepsilon$ whenever $|y| \le 3$ (and t is still fixed). Let

$$P(x) = \int_{x-2}^{x+1} \Psi(y,t) g(x-y)\,\mathrm{d}y = \int_{-1}^{2} \Psi(x-y,t) g(y)\,\mathrm{d}y$$

P is a polynomial, for $\Psi(x-y,t)$ is a polynomial of degree $2n$ in $x-y$, and so differentiation of the above integral more than $2n$ times yields zero. Whenever $x \in [0,1]$ we get $|g(x) - P(x)| \le |g(x) - u(x,t)| + |u(x,t) - P(x)| < \varepsilon + 4\varepsilon \max|g|$ which shows that P can be made to approximate g arbitrarily well in the uniform norm.

#9–1: It is helpful to note that a subharmonic function u (in the present sense) satisfies $u(x) \le \fint_{B(x,r)} u\,\mathrm{d}y$ when $B(x,r) \subset \Omega$, as we can deduce from integration in "polar" coordinates:

$$\fint_{B(x,r)} u\,\mathrm{d}y = \frac{1}{\alpha(n)r^n} \int_0^r \int_{\partial B(x,\rho)} u\,\mathrm{d}S\,\mathrm{d}\rho \ge \frac{1}{\alpha(n)r^n} \int_0^r n\alpha(n)\rho^{n-1} u(x)\,\mathrm{d}\rho = u(x)$$

If u is subharmonic and v is harmonic then $w = u - v$ is subharmonic, since v satisfies the mean value property. Also $w \le 0$ on $\partial\Omega$, and we have to prove that $w \le 0$ in Ω.

Now a subharmonic function satisfies the strong maximum principle, for if w achieves its maximum M at a point $x \in \Omega$ then

$$M = w(x) \le \fint_{B(x,r)} w\,\mathrm{d}y \le M$$

where the second inequality becomes strict if $w < M$ anywhere within $B(x,r)$. That is obviously absurd, so w must equal M in all of $B(x,r)$. It follows, just as in the proof of the maximum principle for harmonic functions, that u must be constant on the component of Ω containing x.

From the strong maximum principle follows the weak maximum principle, that is w achieves its maximum on $\partial\Omega$. Since $w \le 0$ there, $w \le 0$ everywhere — and so we have proved $u \le v$ in Ω.

For the last question, note that if $u \in C^2(\Omega)$ then

$$\varphi(r) = \fint_{\partial B(x,r)} u\,\mathrm{d}S \quad \text{yields} \quad \varphi'(r) = \frac{r}{n} \fint_{B(x,r)} \Delta u\,\mathrm{d}y$$

(see the proof of the mean value formula on p. 30 of Evans' notes.)

Clearly, if $-\Delta u \le 0$ in Ω we get $\varphi' \ge 0$, and since $\varphi(r) \to u(x)$ when $r \searrow 0$, we find $\varphi(r) \ge u(x)$ — and so u is subharmonic.

On the other hand, if $-\Delta u > 0$ at some point in Ω the inequality holds in an open set, since Δu is continuous. But if we place x in this open set, $\varphi'(r) < 0$ for small r, and hence $\varphi(r) < u(x)$ for small r. Hence u is not subharmonic.

We conclude that, for C^2 functions, the new and old definitions of "subharmonic" coincide.

Oppgavesett #10 (til 1997–11–11)

Fra Evans side 84: 20, 22, 23. Det er en tyrkleif i oppgave 20: Det skal stå $\tilde{U} = rU$ (hvor $U(r,t)$ er definert nederst side 68).