75042 Partielle differensialligninger

Oppgavesett #10 (til 1997–11–11) — løsningsforslag

Evans 2.5–20: From $\tilde{U} = rU$ we get $\tilde{U}_{rr} = (rU)_{rr} = (U + rU_r)_r = 2U_r + rU_{rr}$, and of course $\tilde{U}_{tt} = rU_{tt}$.

For $n = 3$, the Euler–Poisson–Darboux equation [(12) on p. 69 in Evans] is $U_{tt} = U_{rr} + 2U_r/r$. Multiplying this by r produces the wave equation $\tilde{U}_{tt} = \tilde{U}_{xx}$.

We are trying to solve $u_{tt} - \Delta u = 0$ in $t > 0$, $x \in \mathbb{R}^3$ with the initial values $u(x,0) = g(x)$ and $u_t(x,0) = h(x)$. By setting $G(x;r) = ⨍_{\partial B(x,r)} g \,\mathrm{d}S$, $H(x;r) = ⨍_{\partial B(x,r)} h \,\mathrm{d}S$, $\tilde{G}(x;r) = rG(x;r)$, and $\tilde{H}(x;r) = rH(x;r)$ we get an initial value problem for $\tilde{U}$ which is solved by the d'Alembert formula:

$$\tilde{U}(x;r,t) = \frac{1}{2}\big(\tilde{G}(x;r+t) + \tilde{G}(x;r-t)\big) + \frac{1}{2}\int_{r-t}^{r+t} \tilde{H}(x;\rho)\,\mathrm{d}\rho.$$

Simply divide by r to get

$$\begin{aligned} U(x;r,t) &= \frac{1}{2}\Big(\frac{r+t}{r}G(x;r+t) + \frac{r-t}{r}G(x;r-t)\Big) + \frac{1}{2r}\int_{r-t}^{r+t} \rho H(x;\rho)\,\mathrm{d}\rho \\ &= \frac{G(x;r+t) + G(x;t-r)}{2} + t\frac{G(x;t+r) - G(x;t-r)}{2r} + \frac{1}{2r}\int_{t-r}^{t+r} \rho H(x;\rho)\,\mathrm{d}\rho \end{aligned}$$

where, in the last integral, I have assumed $0 < r < t$ and used the fact that $H(x;r)$ is an even function of r, so that the integrand is an odd function, and the integral over $[r-t, t-r]$ is zero. I have also used the corresponding evenness of G to replace the argument $r - t$ by $t - r$ in first two terms. It is now a simple matter to let $r \searrow 0$ to get

$$u(x,t) = \lim_{r\to 0} U(x;r,t) = G(x;t) + tG_r(x;t) + tH(x;t).$$

It only remains to compute G_r:

$$\frac{\partial}{\partial r}G(x;r) = \frac{\partial}{\partial r} ⨍_{B(0,1)} g(x+rz)\,\mathrm{d}S(z) = ⨍_{\partial B(0,1)} Dg(x+rz)\cdot z\,\mathrm{d}S(z)$$

which we translate back to an integral over $B(x,r)$ (remembering that $y = x + rz$ implies $z = (y-x)/r$) and we end with

$$u(x,t) = ⨍_{\partial B(x,t)} \big(g(y) + Dg(y)\cdot(y-x) + th(y)\big)\,\mathrm{d}y$$

which is Kirchoff's formula.

Evans 2.5–22: From d'Alembert's formula

$$u(x,t) = \frac{1}{2}\big(g(x+t) + g(x-t)\big) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,\mathrm{d}y$$

we immediately get

$$\begin{aligned} u_t &= \frac{1}{2}\big(g'(x+t) - g'(x-t) + h(x+t) + h(x-t)\big) = F(x+t) - G(x-t), \\ u_x &= \frac{1}{2}\big(g'(x+t) + g'(x-t) + h(x+t) - h(x-t)\big) = F(x+t) + G(x-t) \end{aligned}$$

where

$$F = \frac{1}{2}(g' + h) \quad \text{and} \quad G = \frac{1}{2}(g' - h).$$

Substituting this into the kinetic and potential energies

$$k(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_t^2\,\mathrm{d}x \quad \text{and} \quad p(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_x^2\,\mathrm{d}x$$

and then adding, we get

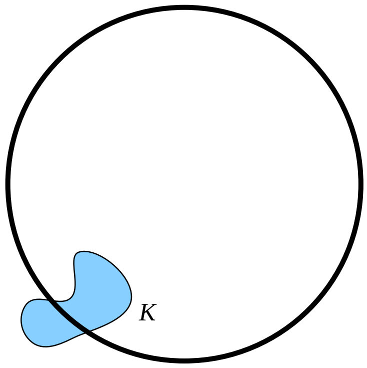
$$k(t)+p(t)=\frac{1}{2}\int_{-\infty}^{\infty}\Big[\big(F(x+t)-G(x-t)\big)^2+\big(F(x+t)+G(x-t)\big)^2\Big]\,\mathrm{d}x$$
$$=\int_{-\infty}^{\infty}F(x+t)^2\,\mathrm{d}x+\int_{-\infty}^{\infty}G(x-t)^2\,\mathrm{d}x=\int_{-\infty}^{\infty}F(x)^2\,\mathrm{d}x+\int_{-\infty}^{\infty}G(x)^2\,\mathrm{d}x$$

which is clearly independent of t. Similarly, subtracting we get

$$k(t)-p(t)=\frac{1}{2}\int_{-\infty}^{\infty}\Big[\big(F(x+t)-G(x-t)\big)^2-\big(F(x+t)+G(x-t)\big)^2\Big]\,\mathrm{d}x$$
$$=-2\int_{-\infty}^{\infty}F(x+t)G(x-t)\,\mathrm{d}x$$

which is zero for large t, since F and G both have compact support and therefore $F(x+t)G(x-t)=0$ for all x when t is sufficiently large.

Evans 2.5–23: We use Kirchoff's formula (see above). Now that formula averages something which appears to grow like t, which doesn't look very promising. However, we have not yet used the compactness of the supports of g and h. This will imply that, when t is large, the joint support K of g and h will only intersect a small portion of $\partial B(x,t)$. More precisely, the surface area of $K\cap\partial B(x,t)$ will be bounded by some number, say A, while the total surface area is proportional to t^2. This should lead to the desired estimate:



$$u(x,t)=\frac{1}{4\pi t^2}\int_{K\cap\partial B(x,t)}\big(g(y)+Dg(y)\cdot(y-x)+th(y)\big)\,\mathrm{d}y$$
$$\le\frac{A}{4\pi t^2}(\max|g|+t\max|Dg|+t\max|h|)\le\frac{C}{t}$$

as was to be proved.

Oppgavesett #11 (til 1997–11–18)

Dette blir siste oppgavesett:

#11–1: Suppose $\Omega\subset\mathbb{R}^n$ is a bounded region with smooth boundary and $u\in C^2(\bar\Omega)$ is harmonic in Ω. Show that

$$\int_{\partial\Omega}u\frac{\partial u}{\partial\nu}\,\mathrm{d}S\ge 0,\qquad\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\,\mathrm{d}S=0.$$

#11–2: Suppose $\Omega\subset\mathbb{R}^n$ is a bounded region with smooth boundary. Study the uniqueness of solutions to the Neumann problem

$$\Delta u=0\text{ in }\Omega,\qquad\frac{\partial u}{\partial\nu}=g\text{ on }\partial\Omega,$$

if a solution exists. Explain why no solution can exist unless the compatibility condition $\int_{\partial\Omega}g\,\mathrm{d}S=0$ is satisfied.

#11–3: Consider "small" waves for the conservation law $u_t+f(u)_x=0$. By "small" I mean that u should be close to a constant u_0. Discuss the behaviour of the solution. What role is played by the characteristic speed $f'(u)$? (The limit of almost constant u is called the "acoustic limit" in analogy with ordinary soundwaves; a very loud sound at 100 dB means pressure variations on the order of 2 Pa, where atmospheric pressure is about 10^5 Pa.)

#11–4: Discuss the solutions of the (rescaled) traffic problem

$$u_t+\big(u(1-u)\big)_x=0$$
$$u(x,0)=\begin{cases}u_1 & 0<x<1\\ u_0 & \text{otherwise}\end{cases}$$

In particular, what is the behaviour of the solution as $t\to\infty$?