# 75042 Partielle differensialligninger

## *Oppgavesett #11 (til 1997–11–18) — løsningsforslag*

**Dette var siste oppgavesett.**

**#11–1:** The equality is almost trivial. This is just the divergence theorem:

$$\int_{\partial\Omega} \frac{\partial u}{\partial\nu}\,\mathrm{d}S = \int_{\partial\Omega} \nu\cdot Du\,\mathrm{d}S = \int_{\Omega} \operatorname{div} Du\,\mathrm{d}x = \int_{\Omega} \Delta u\,\mathrm{d}x = 0$$

The inequality is just partial integration. I remind you that this is nothing but the divergence theorem applied to a product: Since $\operatorname{div}(uDu) = Du\cdot Du + u\operatorname{div} Du = |Du|^2 + u\Delta u = |Du|^2$ (when $u$ is harmonic) we find

$$\int_{\partial\Omega} u\frac{\partial u}{\partial\nu}\,\mathrm{d}S = \int_{\partial\Omega} \nu\cdot uDu\,\mathrm{d}S = \int_{\Omega} \operatorname{div}(uDu)\,\mathrm{d}x = \int_{\Omega} |Du|^2\,\mathrm{d}x \geq 0$$

and, by the way, the inequality must be strict unless $u$ is constant (on each component of $\Omega$).

**#11–2:** We use the linearity. If $u$, $v$ solve the given problem then their difference $w = u - v$ will solve

$$\Delta w = 0 \text{ in } \Omega, \qquad \frac{\partial w}{\partial\nu} = 0 \text{ on } \partial\Omega.$$

Clearly, any constant function satisfies this; so the best we can ever hope to prove is that two solutions of the original problem differ by a constant. And we can indeed prove this, for if $w$ is not constant then by the final remark in the solution of #11–1 above, we have

$$\int_{\partial\Omega} w\frac{\partial w}{\partial\nu}\,\mathrm{d}S > 0$$

which is clearly impossible since $\partial w/\partial\nu = 0$ on $\partial\Omega$. (One sometimes comes by mixed boundary value problems, in which $u$ is given on part of the boundary and $\partial w/\partial\nu$ on the rest. Clearly the same proof works there, and in this case we do get uniqueness, provided only that $u$ is given on a nonempty subset of the boundary.)

The compatibility condition $\int_{\partial\Omega} g\,\mathrm{d}S = 0$ is necessary for the existence of a solution because of the second formula of #11–1.

**#11–3:** Write a "small" wave as $u(x,t) = u_0 + v(x,t)$ where $v$ is small. The equation $u_t + f(u)_x = 0$ in quasilinear form $u_t + f'(u)u_x = 0$ is then equivalent to $v_t + f'(u_0+v)u_x = 0$, and it seems reasonable to replace $f'(u_0+v)$ by $a = f'(u_0)$. Thus we are led to the transport equation $v_t + av_x = 0$, whose general solution is $v = g(x-at)$ where $g$ is an arbitrary function. "Small" waves, therefore, travel with a constant speed $f'(u_0)$ corresponding to the sound speed in acoustics.

The above discussion may be made more complete by considering discontinuous functions $v$. A shock in $v$, as long as $|v|$ itself is small, is just a small shock in $u$. If the left and right states $u_\pm$ are both close to $u_0$ the Rankine-Hugoniot condition becomes

$$\sigma = \frac{[f(u)]}{[u]} = \frac{f(u_+) - f(u_-)}{u_+ - u_-} = f'(\tilde{u}) \approx f'(u_0)$$

where $\tilde{u}$ is between $u_+$ and $u_-$. Thus shocks, too, move with the approximate speed of $f'(u_0)$.

**#11–4:** This is a conservation law with flux function $F(u) = u(1-u) = u - u^2$ and characteristic speed $F'(u) = 1-2u$. The flux function is concave, so the solution to a Riemann problem with left state $u_l$ and right state $u_r$ is a expansion wave if $u_l > u_r$ (since then $F'(u)$ increases as $u$ moves from $u_l$ towards $u_r$) and a shock wave if $u_r > u_l$.

The given initial value problem is just like two Riemann problems, at least for small $t$. There is one at the discontinuity at $x = 0$, and one at $x = 1$ (a Riemann problem with an initial discontinuity at $x = 1$ is completely equivalent to the corresponding problem with the discontinuity at $x = 0$, since replacing $u$ by $u(x-1, t)$ gives us a new solution of the conservation law but with the initial discontinuity moved.)

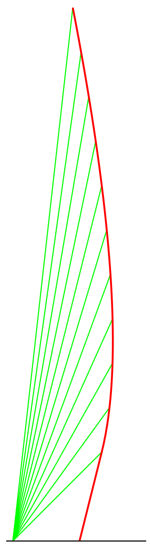
We get to consider two cases:

- $u_1 < u_0$ — expansion wave at $x = 0$, shock at $x = 1$;
- $u_1 > u_0$ — shock at $x = 0$, expansion wave at $x = 1$.

in each case the shock speed is

$$\sigma = \frac{[u-u^2]}{[u]} = \frac{u_1 - u_1^2 - u_0 + u_0^2}{u_1 - u_0} = 1 - u_1 - u_0$$

which is the mean value of the two characteristic speeds $F'(u_\pm)$ (as is always the case with a quadratic flux function). In either case, therefore, the shock will sooner or later run into the expansion wave. The shock will from then on have a constant state on one side and a variable state on the other side, and so the shock speed will change.



We consider in detail only the first case ($u_1 < u_0$). The solution is shown in the margin (for $u_1 = 0$ and $u_0 = 3/4$). Only part of the expansion wave is shown; to the righthand side of the shock, the characteristic speed is $-1/2$.

In the expansion wave, $F'(u) = 1 - 2u = x/t$ so that $u = \frac{1}{2}(1 - x/t)$. The righthand edge of the expansion wave is the characteristic $x = (1 - 2u_1)t$. The shock has speed $1 - u_0 - u_1$ and therefore equation $x = 1 + (1 - u_0 - u_1)t$. The two lines meet at $(x_0, t_0)$:

$$x_0 = \frac{1 - 2u_1}{u_0 - u_1}, \quad t_0 = \frac{1}{u_0 - u_1},$$

so that point marks the beginning of the curved part of the shock, which must satisfy the equation (the Rankine–Hugoniot relation)

$$\dot{s} = 1 - u_0 - \frac{1}{2}\Big(1 - \frac{s}{t}\Big) = \frac{1}{2} - u_0 + \frac{s}{2t}.$$

This equation is easily solved: Solve the corresponding homogeneous equation $\dot{s} = s/(2t)$ first to find $s = C\sqrt{t}$, and then try $s(t) = q(t)\sqrt{t}$ etc. I admit I got lazy and used Maple to solve the equation together with the initial data $s(t_0) = x_0$, and found the solution

$$s = (1 - 2u_0)t + 2\sqrt{(u_0 - u_1)t}.$$

Note that $\lim_{t\to\infty} \dot{s}(t) = 1 - 2u_0$ which is the characteristic speed on the right side of the shock. Hence the characteristic speed on the left side has the same limit (the shock speed being the average of the two characteristic speeds), and the value of $u$ on the left side approaches $u_0$. This means that the strength of the shock approaches zero as $t \to \infty$. As the left edge of the expansion wave is at $x = (1 - 2u_0)t$ the width of the wave is $2\sqrt{(u_0 - u_1)t}$ which grows without bound as $t \to \infty$. For large $t$, therefore, the solution is a very wide wave with values near the constant $u_0$ and a weak shock at the right side.

The other case ($u1 > u_0$) is similar, and can in fact be reduced to the former case by considering $w(x, t) = 1 - u(1 - x, t)$.