Two simple proofs in PDE theory

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Abstract. The proof of Liouville's theorem in Evans' lecture notes on PDEs is based on a somewhat difficult estimate for the higher derivatives of a harmonic function in terms of the values of the function on a ball (Theorem 7, page 32). However, the necessary estimate is much easier to derive directly. The estimate used in the following simple proof is in fact part of the proof in Evans' notes, but we extract that part here for easier reading.

We also give a simple proof of the maximum principle for the heat equation, avoiding the mean value theorem.

1 Theorem. (Liouville) *A function which is bounded and harmonic on $\mathbb{R}^n$ is constant.*

Proof: If u is any smooth function, the divergence theorem applied to the vector field ue_i becomes[1]

$$\int_\Omega u_{x_i}\,\mathrm{d}x = \int_{\partial\Omega} u\nu_i\,\mathrm{d}S.$$

Now, assume u is harmonic and bounded in $\mathbb{R}^n$, say $|u| \le M$ everywhere. We know that u_{x_i} is also harmonic, and so the mean value theorem applied to u_{x_i} becomes

$$u_{x_i}(z) = \frac{1}{\alpha(n)r^n}\int_{B(z,r)} u_{x_i}\,\mathrm{d}x = \frac{1}{\alpha(n)r^n}\int_{\partial B(z,r)} u\nu_i\,\mathrm{d}S.$$

But $|u\nu_i| \le M$, and the area of $\partial B(z,r)$ is $n\alpha(n)r^{n-1}$. Thus we get

$$|u_{x_i}(z)| \le \frac{nM}{r},$$

and since r can be any positive number we have $u_{x_i} = 0$ (for $i = 1,\dots,n$). It follows that u is constant. ∎

Something to think about. The conclusions of Liouville's theorem do not hold if u is merely assumed bounded and harmonic on some open subset of $\mathbb{R}^n$. What part of the above proof would fail in this more general case?

[1] Here, e_i is the i–th coordinate vector: $u_1 = (1,0,\dots,0)$ etc. Also, ν_i is the i–th component of the outward unit normal vector.

In the rest of this note, we shall assume that Ω is a bounded open subset of $\mathbb{R}^n$. As in Evans' notes, $\Omega_T = \Omega \times (0,T)$, and Γ_T is the parabolic boundary of Ω_T.

2 Lemma. *Assume $v \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega_T})$ and $v_t < \Delta v$ in $\Omega \times (0,T]$. Then*

$$\max_{\Omega_T} v = \max_{\Gamma_T} v.$$

Proof: If this is not so, then v has its maximum at a point (x,t) with $x \in \Omega$ and $0 < t \le T$. At such a maximum $v_t \ge 0$ ($v_t > 0$ is possible only if $t = T$), and $\Delta v \le 0$. This contradicts the assumed inequality. ∎

3 Theorem. *Assume $u \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega_T})$ and $u_t \le \Delta u$ in $\Omega \times (0,T]$. Then*

$$\max_{\Omega_T} u = \max_{\Gamma_T} u.$$

Proof: Put $v(x,t) = u(x,t) + \varepsilon|x|^2$. Then $v_t - \Delta v = u_t - \Delta u - 2n\varepsilon < 0$, and therefore v satisfies the lemma when $\varepsilon > 0$. We get (assuming R is big enough so that $\Omega \subseteq B(0,R)$)

$$\max_{\Omega_T} u \le \max_{\Omega_T} v = \max_{\Gamma_T} v \le \max_{\Gamma_T} u + \varepsilon R^2.$$

Since this is so for any $\varepsilon > 0$, the inequality $\max_{\Omega_T} u \le \max_{\Gamma_T} u$ follows. The opposite inequality is, of course, trivial. ∎

The above theorem will of course hold for solutions of the heat equation $u_t = \Delta u$. Solutions of the inequality $u_t \le \Delta u$ are sometimes called *subsolutions* of the heat equation. The theorem is the maximum principle for such subsolutions. They have roughly the same relation to solutions of the heat equation as the subharmonic functions have to harmonic functions.

The maximum principle for (sub)harmonic functions can, of course, be proved using essentially the same technique as used here.