Entropy conditions for conservation laws

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Abstract. These notes are meant as a supplement to the material on conservation laws found in Evans' lecture notes on PDEs.

They are written in English because we often have visiting foreign students attending the course.

If you find misprints or other mistakes or shortcomings of these notes, I would like to hear about it — preferably by email.

The method of vanishing viscosity

These notes deal with the conservation law

$$u_t + F(u)_x = 0 \tag{1}$$

and the problem of determining which weak solutions of (1) are to be considered acceptable. We shall use the idea of *vanishing viscosity*, whereby one considers instead the equation

$$u_t^\varepsilon + F(u^\varepsilon)_x = \varepsilon u_{xx}^\varepsilon \tag{2}$$

in the limit as ε decreases to zero.

In these notes, we shall consider a jump discontinuity in the solution u with left state u_l and right state u_r and shock speed σ, satisfying the Rankine-Hugoniot relation

$$\sigma = \frac{F(u_r) - F(u_l)}{u_r - u_l} \tag{3}$$

and the problem is to determine further conditions on the shock that will, hopefully, lead to uniqueness as well of existence of weak solutions to the initial value problem for (1).

Traveling waves

One way to get an answer to our problem is to look for *traveling wave* solutions to (2), i.e., solutions on the form

$$u^\varepsilon(x,t) = w\Big(\frac{x-\sigma t}{\varepsilon}\Big). \tag{4}$$

Substituting this into (2) we get, after a little tidying up,

$$-\sigma w' + F(w)' = w''$$

where ε does not occur in the equation at all. We can integrate this once to get

$$w'(\xi) = F(w(\xi)) - \sigma w(\xi) - C. \tag{5}$$

Now, we get

$$\lim_{\varepsilon \searrow 0} u(x,t) = \begin{cases} \lim_{\xi\to-\infty} w(\xi), & x < \sigma t, \\ \lim_{\xi\to\infty} w(\xi), & x > \sigma t \end{cases}$$

and therefore we shall insist that

$$\begin{aligned} \lim_{\xi\to-\infty} w(\xi) &= u_l, \\ \lim_{\xi\to\infty} w(\xi) &= u_r. \end{aligned} \tag{6}$$

In particular, the right hand side of (5) has a limit as $\xi \to \pm\infty$, and so $w'(\xi)$ has a limit as well. If this limit is nonzero, then of course w itself cannot have a limit, so in fact the limit must be zero. Thus we get

$$F(u_l) - \sigma u_l - C = 0 \quad \text{and} \quad F(u_r) - \sigma u_r - C = 0 \tag{7}$$

and if we subtract these two equations from each other, we get the Rankine-Hugoniot relation (3).

Now we solve (5) in the usual way as a separable equation, leading to the solution

$$\int \frac{\mathrm{d}w}{F(w) - \sigma w - C}\,\mathrm{d}w = \int \mathrm{d}\xi = \xi \tag{8}$$

(where I chose to ignore the constant of integration on the right hand side). If we want to invert this to express w as a function of ξ, the left hand side of (8) must be a monotone function of w. This means that the sign of $F(w) - \sigma w - C$ must be constant.

Now (7) states that $z = \sigma w + C$ is the equation of the straight line joining the two points $(u_l, F(u_l))$ and $(u_r, F(u_r))$ on the graph of F — that is, the secant between those two points. So we have discovered that the graph $z = F(w)$ must lie to one side of that secant for w between u_l and u_r.

1 Proposition. *The equation* (2) *admits a traveling wave solution of the form* (4) *satisfying* (6) *if and only if*

$$(u_r - u_l)\big(F(w) - \sigma w - C\big) > 0 \quad \textit{for all } w \textit{ between } u_l \textit{ and } u_r$$

where σ and C are chosen so that the above expression is zero when $w = u_l$ or $w = u_r$.

Proof: We have proven most of this already. It only remains to note that if $u_l < u_r$ then w needs to be an increasing function of ξ (we already know it must be monotone), since otherwise we cannot satisfy (6). By (8) this means that $F(w) - \sigma w - C > 0$ for w between u_l and u_r. Similarly, if $u_l > u_r$ then w needs to be a decreasing function of ξ so that $F(w) - \sigma w - C < 0$ instead. These two cases are both included in the inequality stated above. ∎

2 Exercise. Show that the entropy condition given by the above proposition is equivalent to the inequalities

$$\frac{F(w) - F(u_l)}{w - u_l} > \sigma > \frac{F(w) - F(u_r)}{w - u_r}$$

whenever w is between u_l and u_r. This entropy condition was suggested by O. Oleinik, but since she gave another entropy condition more commonly associated with her name this condition is often referred to as the *viscous profile* entropy condition.

3 Exercise. The *Lax entropy condition* is the pair of inequalities

$$F'(u_l) \ge \sigma \ge F'(u_r).$$

Show that this is equivalent with the viscous profile entropy condition in the case when F is either strictly convex or strictly concave. Of course, the Lax condition has the advantage that it is very easy to check.

4 Exercise. Compute the traveling wave solutions of (2) for Burgers' equation (given by $F(u) = u^2/2$). Such a traveling wave solution is often called a *viscous profile* for the limiting shock.

Entropy functions

We shall show, under reasonable conditions, that if there are solutions of (2) converging to a solution of (1) then the shocks in the latter solution must satisfy the entropy condition given above. The proof will be indirect, using yet another entropy condition which we shall now present.

Given a function H, the quantity $H(u)$ can be seen to satisfy a conservation law similar to (1), namely

$$H(u)_t + G(u)_x = 0 \tag{9}$$

if u is a differentiable function satisfying (1). Indeed, when u is smooth (1) can be written in quasilinear form as $u_t + F'(u)u_x = 0$, while (9) becomes $H'(u)u_t + G'(u)u_x = 0$, which will be satisfied if

$$G'(u) = H'(u)F'(u).$$

Such a pair of functions H, G will be called an *entropy pair*. If u is a weak solution of (1) with a shock, however, it will not in general be a weak solution of (9). But if H is a convex function, (9) may be replaced by the inequality

$$H(u)_t + G(u)_x \le 0 \tag{10}$$

which is to hold in the weak sense, meaning

$$\iint \big(H(u)\varphi_t + G(u)\varphi_x\big)\,\mathrm{d}x\mathrm{d}t \ge 0 \tag{11}$$

whenever $\varphi \geq 0$ is a smooth test function. (The inequality is reversed with respect to the previous equation because of the minus sign introduced by a partial integration.) This is called the *entropy inequality*, and is the alternative entropy condition we shall derive.

5 Theorem. *Assume* (2) *has solutions* u^ε *for each* $\varepsilon > 0$ *which converge to a solution* u *of* (1) *as* $\varepsilon \searrow 0$ *in the sense that*

$$\iint |u^\varepsilon - u| \,\mathrm{d}x\mathrm{d}t \to 0 \quad (\varepsilon \searrow 0).$$

Assume also that there is a constant M *so that* $|u| \leq M$ *and* $|u^\varepsilon| \leq M$ *for all* $\varepsilon > 0$. *Let* H, G *be an entropy pair with* H *convex. Then the entropy inequality* (11) *holds.*

Proof: We begin with a look at the approximation to the left hand side of (10):

$$\begin{aligned} H(u^\varepsilon)_t + G(u^\varepsilon)_x &= H' u^\varepsilon_t + G' u^\varepsilon_x = H' \cdot (u^\varepsilon_t + F' u^\varepsilon_x) = \varepsilon H' \cdot u^\varepsilon_{xx} \\ &= \varepsilon\big(H(u^\varepsilon)_{xx} - H''(u^\varepsilon)\cdot(u^\varepsilon_x)^2\big) \leq \varepsilon H(u^\varepsilon)_{xx} \end{aligned}$$

where the last equality is based on the calculation

$$\begin{aligned} H(u^\varepsilon)_x &= H'(u^\varepsilon)u^\varepsilon_x, \\ H(u^\varepsilon)_{xx} &= \big(H'(u^\varepsilon)\big)_x u^\varepsilon_x + H'(u^\varepsilon)u^\varepsilon_{xx} = H''(u^\varepsilon)\cdot(u^\varepsilon_x)^2 + H'(u^\varepsilon)u^\varepsilon_{xx} \end{aligned}$$

and the final inequality holds because H is convex, so $H'' \geq 0$.

We now turn our attention to the approximated left hand side of (11).

$$\begin{aligned} \iint \big(H(u^\varepsilon)\varphi_t + G(u^\varepsilon)\varphi_x\big)\,\mathrm{d}x\mathrm{d}t &= -\iint \big(H(u^\varepsilon)_t + G(u^\varepsilon)_x\big)\varphi \,\mathrm{d}x\mathrm{d}t \\ &\geq -\varepsilon \iint H(u^\varepsilon)_{xx}\varphi\,\mathrm{d}x\mathrm{d}t \\ &= -\varepsilon \iint H(u^\varepsilon)\varphi_{xx}\,\mathrm{d}x\mathrm{d}t \to 0 \quad (\varepsilon \searrow 0) \end{aligned}$$

and to complete the proof we only need to show that the left hand side of the above converges to that of (11), as ε decreases to 0.

$$\begin{aligned} &\left|\iint H(u^\varepsilon)\varphi_t\,\mathrm{d}x\mathrm{d}t - \iint H(u)\varphi_t\,\mathrm{d}x\mathrm{d}t\right| \\ &\leq \iint |H(u^\varepsilon) - H(u)|\cdot|\varphi_t|\,\mathrm{d}x\mathrm{d}t \\ &\leq C(\max|\varphi_t|)\iint |u^\varepsilon - u|\,\mathrm{d}x\mathrm{d}t \to 0 \quad (\varepsilon \searrow 0) \end{aligned}$$

and similarly $\iint G(u^\varepsilon)\varphi_x\,\mathrm{d}x\mathrm{d}t \to \iint G(u)\varphi_x\,\mathrm{d}x\mathrm{d}t$, which completes the proof. ∎

6 Lemma. *Under the conditions of Theorem 5, given any isolated shock of the solution* u *on which* u *has limits on both sides of the shock, the limit values on the two sides of the shock must satisfy the inequality*

$$G(u_r) - G(u_l) \leq \sigma\big(H(u_r) - H(u_l)\big)$$

Proof: We concentrate on a small region of the shock. Let Ω be a neighbourhood of a point on the shock, and let $\varphi \geq 0$ be a test function vanishing outside Ω. Now use (11) and split the region Ω into one part Ω_l to the left of the shock and the other part Ω_r on the right side. The computation below is based on a partial integration (using Green's theorem)[1] and the relation $\mathrm{d}x = \sigma\mathrm{d}t$ along the shock curve.

$$\begin{aligned} 0 &\leq \iint_{\Omega_l} \big(H(u)\varphi_t + G(u)\varphi_x\big)\,\mathrm{d}x\mathrm{d}t + \iint_{\Omega_r} \big(H(u)\varphi_t + G(u)\varphi_x\big)\,\mathrm{d}x\mathrm{d}t \\ &= -\iint_{\Omega_l} \underbrace{\big(H(u)_t + G(u)_x\big)}_{=0}\varphi\,\mathrm{d}x\mathrm{d}t - \iint_{\Omega_r} \underbrace{\big(H(u)_t + G(u)_x\big)}_{=0}\varphi\,\mathrm{d}x\mathrm{d}t \\ &\quad + \int_{\partial\Omega_l} \varphi\cdot(-H(u)\mathrm{d}x + G(u)\mathrm{d}t) + \int_{\partial\Omega_r} \varphi\cdot(-H(u)\mathrm{d}x + G(u)\mathrm{d}t) \\ &= \int_{t_1}^{t_2} \varphi\cdot(-\sigma H(u_l) + G(u_l))\mathrm{d}t - \int_{t_1}^{t_2} \varphi\cdot(-\sigma H(u_r) + G(u_r))\mathrm{d}t \\ &= -\int_{t_1}^{t_2} \big(G(u_r) - G(u_l) - \sigma\big(H(u_r) - H(u_l)\big)\big)\varphi\,\mathrm{d}t \end{aligned}$$

[1] See the Appendix.

and since this must hold for every nonnegative test function φ, the inequality stated in the lemma must hold. ∎

7 Exercise. Show that the inequality of the above lemma implies the Lax entropy condition in the special case of Burgers' equation ($H(u) = u^2/2$).

8 Theorem. *Under the conditions of Theorem 5, any isolated shock of the solution u on which u has limits on both sides of the shock must satisfy the entropy condition of Proposition 1.*

Proof: The trick is to rewrite the inequality of Lemma 6 as an integral by writing the differences as integrals:

$$\begin{aligned}0 &\le \sigma\bigl(H(u_r) - H(u_l)\bigr) - \bigl(G(u_r) - G(u_l)\bigr)\\ &= \int_{u_l}^{u_r} \bigl(\sigma H'(w) - G'(w)\bigr)\,\mathrm{d}w\\ &= \int_{u_l}^{u_r} \bigl(\sigma - F'(w)\bigr)H'(w)\,\mathrm{d}w\\ &= \int_{u_l}^{u_r} \bigl(\sigma w + C - F(w)\bigr)H''(w)\,\mathrm{d}w\end{aligned}$$

where the last step is a partial integration with the constant C chosen according to (7), so that the boundary terms vanish. But this holds for *every* smooth convex function H, which means that H'' is an arbitrary nonnegative function. From the inequality version of the duBois–Reymond lemma it follows that $\sigma w + C - F(w)$ has the same sign as $u_r - u_l$ always, and this completes the proof. ∎

Appendix: Partial integration using Green's theorem

Partial integration, in all its different forms, is a much used and important tool in PDE theory, as should be clear from these notes.

Recall that the classical result on partial integration is just the result of putting $f = gh$ in the fundamental theorem of calculus $\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a)$ and expanding $f' = g'h + gh'$.

The theorems of Stokes and Gauss have the same basic structure as the fundamental theorem of calculus, with the integral being over some higher dimensional set, f' being replaced by a different kind of differentiation (curl og divergence), and $f(b) - f(a)$ being replaced by a boundary integral. Hence these theorems give rise to rules of partial integration by the procedure outlined in the previous paragraph.

Here we carry this out for Green's theorem, which is of the form

$$\iint_\Omega \Bigl(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial t}\Bigr) = \int_{\partial\Omega} (P\,\mathrm{d}x + Q\,\mathrm{d}t).$$

If we replace P and Q by products, for example $Q = G\varphi$ and $P = -H\varphi$, we get

$$\iint_\Omega (G\varphi_x + G_x\varphi + H\varphi_t + H_t\varphi)\,\mathrm{d}x\mathrm{d}t = \int_{\partial\Omega} (-H\,\mathrm{d}x + G\,\mathrm{d}t)$$

which, when rearranged into

$$\iint_\Omega (G\varphi_x + H\varphi_t)\,\mathrm{d}x\mathrm{d}t = \int_{\partial\Omega} (-H\,\mathrm{d}x + G\,\mathrm{d}t) - \iint_\Omega (G_x + H_t)\varphi\,\mathrm{d}x\mathrm{d}t$$

is a version of partial integration deduced from Green's theorem as used in the proof of Lemma 6.